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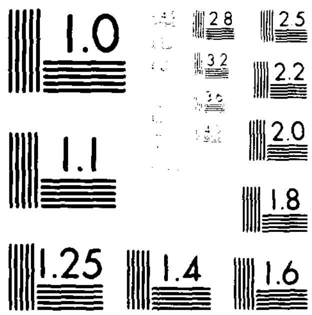

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Extremal and Related Properties of Stationary Processes
Part II: Extreme Values in Continuous Time

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PART II

EXTREME VALUES IN CONTINUOUS TIME

In this part of the work we shall explore extremal and related theory for continuous parameter stationary processes. As we shall see (in Chapter 13) it is possible to obtain a satisfying general theory extending that for the sequence case, described in Chapter 2 of Part I, and based on dependence conditions closely related to those used there for sequences. In particular, a general form of Gnedenko's Theorem will be obtained for the maximum

$$M(T) = \sup\{\xi(t);\ 0 \le t \le T\}$$

where $\xi(t)$ is a stationary stochastic process satisfying appropriate regularity and dependence conditions.

Before presenting this general theory, however, we shall give a detailed development for the case of stationary normal processes, for which very many explicit extremal and related results are known. For mean-square differentiable normal processes, it is illuminating and profitable to approach extremal theory through a consideration of the properties of *upcrossings* of a high level (which are analogous to the exceedances used in the discrete case). The basic framework and resulting extremal results are described in Chapters 6 and 7 respectively.

As a result of this limit theory it is possible to show that the point process of upcrossings of a level takes on an increasingly Poisson character as the level becomes higher. This and related properties are discussed in Chapter 8, and are analogous to the corresponding results for exceedances by stationary normal sequences, given in Chapter 4.

The location of the maximum (primarily under normality) is considered in Chapter 9, along with a derivation of asymptotic Poisson properties for the point process in the plane given by the locations and heights of all *local* maxima. The latter results provice asymptotic joint distributions for the locations and heights of any given number of the largest local maxima.

The local behavior of a stationary normal process near a high level upcrossing is discussed in Chapter 10, using, in particular, a simple process (the "Slepian model process") to describe the sample paths at such an upcrossing. As an interesting corollary it is possible to obtain the limiting distribution for the lengths of excursions by stationary normal processes above a high level, under appropriate conditions.

In Chapter 11 we consider the joint asymptotic behavior of the maximum and minimum of a stationary normal process, and of maxima of two or more dependent processes. In particular it is shown that - short of perfect correlation between the processes - such maxima are asymptotically independent.

While the mean square differentiable stationary normal processes form a substantial class, there are important stationary normal processes (such as the Ornstein-Uhlenbeck process) which do not possess this property, Many of these have covariance functions of the form $r(\tau) = 1 - C|\tau|^{\alpha} + o|\tau|^{\alpha}$ as $\tau \to 0$ for some α, $0 < \alpha < 2$ (the case $\alpha = 2$ corresponds to the mean-square differentiable processes). The extremal theory for these processes is developed in Chapter 12, using more sophisticated methods than those of Chapter 7, for which simple considerations involving upcrossings sufficed.

Finally, Chapter 13 contains the promised general extremal theory (including Gnedenko's Theorem) for stationary continuous-time processes which are not necessarily normal. This theory essentially relies on the discrete parameter results of Part I, by means of the simple device of expressing the maximum of a continuous parameter process in say time $T = n$, an integer, as the maximum of n "submaxima", over fixed intervals, viz.

$$M(n) = \max(\zeta_1, \zeta_2, \ldots, \zeta_n)$$

where $\zeta_i = \sup\{\xi(t);\ i-1 \leq t \leq i\}$. It should be noted (as shown in Chapter 13) that the results for stationary normal processes given in

Chapters 7 and 12 can be obtained from those in Chapter 13 by specialization. However, since most of the effort required in Chapters 7 and 12 is still needed to verify the general conditions of Chapter 13, and the normal case is particularly important, we have felt it desirable and helpful to first treat normal cases separately.

CHAPTER 6

BASIC PROPERTIES OF EXTREMES AND LEVEL CROSSINGS

We turn our attention now to *continuous parameter* stationary processes. We shall be especially concerned with stationary *normal* processes in this and most of the subsequent chapters but begin with a discussion of some basic properties which are relevant whether or not the process is normal, and which will be useful in the discussion of extremal behaviour in later chapters.

We shall consider a stationary process $\{\xi(t); t \geq 0\}$ having a continuous ("time") parameter $t \geq 0$. Stationarity is to be taken in the *strict* sense, i.e. to mean that any group $\xi(t_1), \ldots, \xi(t_n)$ has the same distribution as $\xi(t_1+\tau), \ldots, \xi(t_n+\tau)$ for all τ. Equivalently this means that the *finite dimensional distributions* $F_{t_1,\ldots,t_n}(x_1,\ldots,x_n) = P\{\xi(t_1) \leq x_1,\ldots,\xi(t_n) \leq x_n\}$ are such that $F_{t_1+\tau,\ldots,t_n+\tau} = F_{t_1,\ldots,t_n}$ for all choices of τ, n, and $t_1, t_2, \ldots, t_n$.

It will be assumed throughout without comment that for each t, the d.f. $F_t(x)$ of $\xi(t)$ is continuous. It will further be assumed that, with probability one, $\xi(t)$ has continuous sample functions — that is the functions $\{\xi(t)\}$ are a.s. continuous as functions of $t \geq 0$.

Finally we shall assume that the basic underlying probability measure space has been *completed*, if not already complete. This means in particular that probability-one limits of r.v.'s will themselves be r.v.'s – a fact which will be useful below.

A principal aim in later chapters will be to discuss the behaviour of the maximum

$$M(T) = \sup\{\xi(t); 0 \leq t \leq T\}$$

(which is well defined and attained, since ξ is continuous) especially when T becomes large. It is often convenient to approximate the process $\xi(t)$ by a sequence $\{\xi_n(t)\}$ of processes taking the value $\xi(t)$ at all points of the form jq_n, $j = 0, 1, 2, \ldots$, and being linear between such points, where $q_n \downarrow 0$ as $n \to \infty$. In particular this is useful in showing that $M(T)$ is a r.v., as the following small result demonstrates.

LEMMA 6.1 With the above notation, suppose that $q_n \downarrow 0$ and write $M_n(T) = \max\{\xi(jq_n);\ 0 \le jq_n \le T\}$. Then $M_n(T) \to M(T)$ a.s. as $n \to \infty$, and $M(T)$ is a r.v.

PROOF $M_n(T)$ is the maximum of a finite number of r.v.'s and hence is a r.v. for each n. It is clear from a.s. continuity of $\xi(t)$ that $M_n(T) \to M(T)$ a.s. and hence by completeness $M(T)$ is a r.v. □

We shall also use the notation $M(I)$ to denote the supremum of $\xi(t)$ in any given interval I – of course it may be similarly shown that $M(I)$ is a r.v.

Level crossings and their basic properties

In the discussion of maxima of sequences in Part I, exceedances of a level played an important role. In the continuous case a corresponding role is played by the *upcrossings* of a level for which analogous results (such as Poisson limits) may be obtained. To discuss upcrossings, it will be convenient to introduce – for any real u – a class G_u of all functions f which are continuous on the positive real line, and not identically equal to u in any subinterval. It is easy to see that the sample paths of our stationary process $\xi(t)$ are, with probability one, members of G_u. In fact, every interval contains at least one rational point, and hence

$$P\{\xi(\cdot) \notin G_u\} \le \sum_{j=1}^{\infty} P\{\xi(t_j) = u\},$$

where $\{t_j\}$ is an enumeration of the rational points. Since $\xi(t_j)$ has a continuous distribution by assumption, $P\{\xi(t_j) = u\}$ is zero for every j.

We shall say that the function $f \in G_u$ has a *strict upcrossing* of u

at the point $t_0 > 0$ if for some $\varepsilon > 0$, $f(t) \leq u$ in the interval $(t_0 - \varepsilon, t_0)$ and $f(t) \geq u$ in $(t_0, t_0 + \varepsilon)$. The continuity of f requires, of course, that $f(t_0) = u$, and the definition of G_u that $f(t) < u$ at some points $t \in (t_0 - \eta, t_0)$ and $f(t) > u$ at some points $t \in (t_0, t_0 + \eta)$ for each $\eta > 0$.

It will be convenient to enlarge this notion slightly to include also some points as upcrossings where the behaviour of f is less regular. As we shall see, these further points will not appear in practice for the processes considered in the next two chapters, but are useful in our calculations and will often actually occur for the less regular processes of Chapter 12. Specifically we shall say that the function $f \in G_u$ has an *upcrossing* at $t_0 > 0$ if for some $\varepsilon > 0$ and all $\eta > 0$, $f(t) \leq u$ for all t in $(t_0 - \varepsilon, t_0)$ and $f(t) > u$ for *some* t (and hence infinitely many t) in $(t_0, t_0 + \eta)$. An example of a non-strict upcrossing of zero at t_0 is provided by the function $f(t) = t - t_0$ for $t \leq t_0$ and $f(t) = (t - t_0)\sin((t - t_0)^{-1})$ for $t > t_0$.

The following result contains basic simple facts which we shall need in counting upcrossings.

<u>LEMMA 6.2</u> Let $f \in G_u$ for some fixed u. Then,

(i) if for fixed t_1, t_2, $0 < t_1 < t_2$, we have $f(t_1) < u < f(t_2)$, then f has an upcrossing (not necessarily strict) of u somewhere in (t_1, t_2),

(ii) if f has an upcrossing of u at t_0 which is *not* strict, it has infinitely many upcrossings of u in $(t_0, t_0 + \varepsilon)$, for any $\varepsilon > 0$.

<u>PROOF</u> (i) If $f(t_1) < u < f(t_2)$ with $t_1 < t_2$ write

$$t_0 = \sup\{t > t_1;\ f(s) \leq u \text{ for all } t_1 \leq s \leq t\}.$$

Clearly $t_1 < t_0 < t_2$ and t_0 is an upcrossing point of u by f.

(ii) If t_0 is an upcrossing point of u by f and $\varepsilon > 0$, there is certainly a point t_2 in the interval $(t_0, t_0 + \varepsilon)$ with $f(t_2) > u$. If

t_0 is not a strict upcrossing there must be a point t_1 in (t_0, t_2) such that $f(t_1) < u$. By (i) there is an upcrossing between t_1 and t_2, so that (ii) follows, since $\varepsilon > 0$ is arbitrary. □

Downcrossings (strict or otherwise) may be defined by making the obvious changes, and *crossings* as points which are either up- or downcrossings. Clearly at any crossing t_0 of u we have $f(t_0) = u$. On the other hand there may be "u-values" (i.e. points t_0 where $f(t_0) = u$) which are not crossings – such as points where f is tangential to u or points t_0 such that $f(t) - u$ is both positive and negative in every right and left neighbourhood of t_0 – as for the function $u + (t - t_0)\sin((t - t_0)^{-1})$.

The above discussion applies to each sample function of our process $\xi(t)$ satisfying the general conditions stated since, as noted, the sample functions belong to G_u with probability one. Write, now, $N_u(I)$ to denote the number of upcrossings of the level u by $\xi(t)$ in a bounded interval I, and $N_u(t) = N_u((0,t])$. We shall also sometimes write $N(t)$ for $N_u(t)$ when no confusion can arise.

In a similar way to that used for maxima, it is convenient to use the "piecewise linear" approximating processes $\{\xi_n(t)\}$ to show that $N_u(I)$ is a r.v. and, indeed, in subsequent calculations, as for example in obtaining $E(N_u(I))$. This will be seen in the following lemma, where it will be convenient to introduce the notation

(6.1) $\quad J_q(u) = P\{\xi(0) < u < \xi(q)\}/q, \; q > 0.$

<u>LEMMA 6.3</u> Let I be a fixed, bounded interval. With the above general assumptions concerning the stationary process ξ, let $\{q_n\}$ be any sequence such that $q_n \downarrow 0$ and let N_n denote the number of points jq_n, $j = 1, 2, \ldots$ such that both $(j-1)q_n$ and jq_n belong to I, and $\xi((j-1)q_n) < u < \xi(jq_n)$. Then

(i) $N_n \le N_u(I)$,

(ii) $N_n \to N_u(I)$ a.s. as $n \to \infty$ and hence $N_u(I)$ is a (possibly infinite valued) r.v.,

(iii) $E(N_n) \to E(N_u(I))$ and hence $E(N_u(I)) = \lim_{q \downarrow 0} J_q(u)$.

PROOF (i) If for some j, $\xi((j-1)q_n) < u < \xi(jq_n)$ it follows from Lemma 6.2 (i), that ξ has an upcrossing between $(j-1)q_n$ and jq_n so that (i) follows at once.

(ii) Since the distribution of $\xi(jq_n)$ is continuous and the set $\{kq_n;\ k = 0, 1, 2, \ldots;\ n = 1, 2, \ldots\}$ is countable, we see that $P\{\xi(kq_n) = u$ for any $k = 0, 1, 2, \ldots;\ n = 1, 2, \ldots\} = 0$, and hence we may assume that $\xi(kq_n) \neq u$ for any k and n. We may likewise assume that ξ does not take the value u at either endpoint of I and hence that no upcrossings occur at the endpoints. Now, if for an integer m, we have $N_u(I) \ge m$, we may choose m distinct upcrossings $t_1, \ldots, t_m$ of u by $\xi(t)$ in the interior of I which may, by choice of $\varepsilon > 0$, be surrounded by disjoint subintervals $(t_i - \varepsilon, t_i + \varepsilon)$, $i = 1, 2, \ldots, m$, of I, such that $\xi(t) \le u$ in $(t_i - \varepsilon, t_i)$ and $\xi(\tau) > u$ for some $\tau \in (t_i, t_i + \varepsilon)$. By continuity, τ is contained in an interval – which may be taken as a subinterval of I – in which $\xi(t) > u$. For all sufficiently large n this interval must contain a point kq_n.

Thus there are points $\ell q_n \in (t_i - \varepsilon, t_i)$, $kq_n \in (t_i, t_i + \varepsilon)$ such that $\xi(\ell q_n) < u < \xi(kq_n)$. For some j with $\ell < j \le k$ we must thus have $\xi((j-1)q_n) < u < \xi(jq_n)$. Since eventually each interval $(t_i - \varepsilon, t_i + \varepsilon)$ contains such a point jq_n we conclude that $N_n \ge m$ when n is sufficiently large, from which it follows at once that $\liminf_{n\to\infty} N_n \ge N_u(I)$ (finite or not). Since by (i), $\limsup_{n\to\infty} N_n \le N_u(I)$ we see that $\lim_{n\to\infty} N_n = N_u(I)$ as required. Finally it is easily seen that N_n is a r.v. for each n (N_n is a finite sum of r.v.'s $\chi_k = 1$ if $\xi((k-1)q_n) < u < \xi(kq_n)$, and zero otherwise) so that, by completeness, its a.s. limit $N_u(I)$ is also a r.v., though possibly taking infinite values.

(iii) Since $N_n \to N_u(I)$ a.s., Fatou's Lemma shows that $\liminf_{n\to\infty} E(N_n) \geq E(N_u(I))$. If $E(N_u(I)) = \infty$ this shows at once that $E(N_n) \to E(N_u(I))$. But the same result holds, by dominated convergence, if $E(N_u(I)) < \infty$, since $N_n \leq N_u(I)$ and $N_n \to N_u(I)$ a.s.

Finally, if $I = (0,1)$, then I contains $\nu_n \sim q_n^{-1}$ points jq_n so that, using stationarity,

$$E(N_n) = (\nu_n - 1)\ P\{\xi(0) < u < \xi(q_n)\} \sim J_{q_n}(u).$$

Hence $J_{q_n}(u) \to E(N_u(1))$ from which the final conclusion of (iii) follows since the sequence $\{q_n\}$ is arbitrary. □

COROLLARY 6.4 If $E(N_u(I)) < \infty$, or equivalently if $\liminf_{n\to\infty} J_{q_n}(u) < \infty$ for some sequence $q_n \downarrow 0$, then the upcrossings of u are a.s. strict.

PROOF If $E(N_u(I)) < \infty$ then $N_u(I) < \infty$ a.s. and the assertion follows from (ii) of Lemma 6.2. □

In passing, we shall derive two small results concerning the maximum $M(T)$ and the nature of solutions to the equation $\xi(t) = u$, which rely only on the assumption that $E(N_u(1))$ is a continuous function of u.

THEOREM 6.5 Suppose that $E(N_u(1))$ is continuous at the point u and, as usual that $P(\xi(t) = u) = 0$ so that $\xi(\cdot) \in G_u$ with probability one. Then

(i) if $t \in (0,1)$ and $\xi(t) = u$, then with probability one t is either an upcrossing or a downcrossing point,

(ii) the distribution of $M(1)$ is continuous at u, i.e. $P\{M(1) = u\} = 0$.

PROOF (i) If $\xi(t) = u$, but t is neither an upcrossing nor a downcrossing point it is either a tangency from below or above, i.e. for some $\varepsilon > 0$, $\xi(t) \leq u$ ($\geq u$) for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ or else there are infinitely many upcrossings in $(t_0 - \varepsilon, t_0)$, (and this is precluded by the finiteness of $E(N_u(1))$). Further for each fixed u the probability of tangencies of u from below is zero. To see this, let B_u

be the number of such tangencies of the level u in $(0,1)$, and suppose $N_u + B_u \geq m$, so that there are at least m points $t_1, \ldots, t_m$ which are either u-upcrossings or tangencies from below. Since $\xi(\cdot) \in G_u$ with probability one, there is at least one upcrossing of the level $u - 1/n$ just to the left of any t_j, for all sufficiently large n. This implies

$$N_u + B_u \leq \liminf_{n\to\infty} N_{u-1/n},$$

and applying Fatou's Lemma,

$$E(N_u(1)) + E(B_u(1)) \leq \liminf_{n\to\infty} E(N_{u-1/n}(1)) = E(N_u(1))$$

if $E(N_u(1))$ is continuous. Since $B_u \geq 0$, we conclude that $B_u = 0$ (with probability one). A similar argument excludes tangencies from above, and we have proved that all u-values are either up- or downcrossings.

(ii) Since

$$P\{M(1) = u\} \leq P\{\xi(0) = u\} + P\{\xi(1) = u\} + P\{B_u \geq 1\}$$

the result follows as in the proof of part (i) from $P\{B_u = 0\} = 1$. □

Crossings by normal processes

Up to this point we have been considering a quite general stationary process $\{\xi(t);\ t \geq 0\}$. We specialize now to the case of a (stationary) *normal* or *Gaussian* process, by which we mean that the joint distribution of $\xi(t_1), \ldots, \xi(t_n)$ is multivariate normal for each choice of $n = 1, 2, \ldots$ and $t_1, t_2, \ldots, t_n$. It will be assumed without comment that $\xi(t)$ has been standardized to have zero mean and unit variance. The covariance function $r(\tau)$ will then be equal to $E(\xi(t)\xi(t+\tau))$.

Obviously $r(\tau)$ is an even function of τ, with $r(0) = E(\xi^2(t)) = 1$. Thus if r is differentiable at $\tau = 0$, its derivative must be zero there. It is of particular interest to us whether r has *two* derivatives at $\tau = 0$. If $r''(0)$ does exist (finite), it must be negative and we write $\lambda_2 = -r''(0)$. The quantity λ_2 is the second spectral moment, so called since we have $\lambda_2 = \int_{-\infty}^{\infty} \lambda^2 dF(\lambda)$, where $F(\lambda)$ is the spectral d.f., i.e. $r(\tau) = \int_{-\infty}^{\infty} e^{i\lambda\tau} dF(\lambda)$. If r is not twice differentiable at zero then $\int_{-\infty}^{\infty} \lambda^2 dF(\lambda) = \infty$, i.e. $\lambda_2 = \infty$. When $\lambda_2 < \infty$ we have the expansion

$$(6.2)\qquad r(\tau) = 1 - \lambda_2\tau^2/2 + o(\tau^2) \quad \text{as} \quad \tau \to 0.$$

Furthermore, it may be shown that $\lambda_2 = -r''(0) < \infty$ if and only if $\xi(t)$ is differentiable in quadratic mean, i.e. if and only if there is a process $\{\xi'(t)\}$ such that $h^{-1}(\xi(t+h) - \xi(t)) \to \xi'(t)$ in quadratic mean as $h \to 0$, and that then

$$E(\xi'(t)) = 0,\ \mathrm{Var}(\xi'(t)) = -r''(0),$$

$\xi(t)$, $\xi'(t)$ being jointly normal and independent for each I. Furthermore

$$\mathrm{Cov}(\xi'(t),\ \xi'(t+\tau)) = -r''(\tau).$$

For futur se we introduce also

$$\lambda_0 = \int_{-\infty}^{\infty} dF(\lambda) = r(0) = 1 \quad \text{and} \quad \lambda_4 = \int_{-\infty}^{\infty} \lambda^4\, dF(\lambda),$$

where also $\lambda_4 = r^{(4)}(0)$, when finite. An account of these and related properties may be found in Cramér and Leadbetter (1967), Chapter 9.

To apply the general results concerning upcrossings to the normal case we require that $\xi(t)$ should have a.s. continuous sample paths. It is known (cf. Cramér and Leadbetter (1967)), that if

$$(6.3)\qquad 1 - r(\tau) \le C/|\log|\tau||^a \ \text{for some} \ C > 0,\ a > 1, \ \text{for} \ |\tau| < 1,$$

it is possible to define the process $\xi(t)$ as a continuous process. This is a very weak condition which will always hold under assumptions to be used here and subsequently – for example it is certainly guaranteed if r is differentiable at the origin, or even if $1 - r(\tau) \le C|\tau|^\alpha$ for some $\alpha > 0$, $C > 0$.

In the remainder of this and in the next chapters we shall consider a stationary normal process $\xi(t)$, standardized as above, and such that $\lambda_2 < \infty$. To evaluate the mean number of upcrossings of u per unit time we need to evaluate the limit of $J_q(u)$ defined by (6.1) as $q \to 0$.

This is obtained in the following lemma which is a more general result than we need at present, but which will be useful later also.

Let ϕ and Φ denote the standard normal density and distribution functions.

LEMMA 6.6 Let $\{\xi(t)\}$ be a (standardized) stationary normal process with $\lambda_2 < \infty$ and write μ $(= \mu(u)) = \frac{\sqrt{\lambda_2}}{2\pi} e^{-u^2/2}$. Let $q \downarrow 0$ and u either be fixed or tend to infinity as $q \downarrow 0$ in such a way that $uq \to 0$. Then

$$J_q(u) = q^{-1}P\{\xi(0) < u < \xi(q)\} \sim \mu \quad \text{as} \quad q \downarrow 0.$$

PROOF By rewriting the event $\{\xi(0) < u < \xi(q)\}$ as $\{|\xi(0) + \xi(q) - 2u| < \xi(q) - \xi(0)\}$, i.e. as $\{|\zeta_1 - u| < \frac{q}{2}\zeta_2\}$ where $\zeta_1 = (\xi(0) + \xi(q))/2$, $\zeta_2 = (\xi(q) - \xi(0))/q$, are uncorrelated, and hence independent, with respective variances $\sigma_1^2 = (1 + r(q))/2$, $\sigma_2^2 = 2(1 - r(q))/q^2$, we obtain

$$\mu^{-1}J_q(u) = (\mu q \sigma_2)^{-1} \int_{y=0}^{\infty} \phi\left(\frac{y}{\sigma_2}\right) P\{|\zeta_1 - u| < \tfrac{qy}{2}\}\, dy$$

$$= (\mu q \sigma_1 \sigma_2)^{-1} \int_{y=0}^{\infty} \int_{x=u-qy/2}^{u+qy/2} \phi\left(\frac{x}{\sigma_1}\right) \phi\left(\frac{y}{\sigma_2}\right) dx\, dy$$

$$(6.4) \qquad = \int_{y=0}^{\infty} \frac{y}{\sigma_2^2} e^{-y^2/2\sigma_2^2} \left\{\frac{\sigma_2}{2\sigma_1 \mu \sqrt{2\pi}} \int_{x=-1}^{1} \phi\left(\frac{u + qxy/2}{\sigma_1}\right) dx\right\} dy.$$

Now, by simple calculation, the second factor in the integrand may be written as

$$\frac{\sigma_2}{2\sigma_1\sqrt{\lambda_2}} \int_{x=-1}^{1} \exp\left\{-\frac{u^2}{2\sigma_1^2}(1 - \sigma_1^2) - \frac{uqxy}{2\sigma_1^2} - \frac{q^2x^2y^2}{8\sigma_1^2}\right\} dx,$$

which by bounded convergence $(\sigma_1 \to 1,\ 1 - \sigma_1^2 = \lambda_2 q^2/4 + o(q^2),\ \sigma_2 \to \sqrt{\lambda_2})$ tends to 1. It is also immediate that the integrand of (6.4) is dominated by the integrable function Aye^{-cy^2} (for some constants $A, c > 0$) so that an application of dominated convergence gives

$$\lim_{q \to 0} \mu^{-1} J_q(u) = \int_0^{\infty} \frac{y}{\sigma_2^2} e^{-y^2/2\sigma_2^2} dy = 1. \qquad \square$$

The following result – due in its original form to S.O. Rice (1945) – is now an immediate corollary of this lemma.

THEOREM 6.7 (Rice's Formula) If $\{\xi(t)\}$ is a (standardized) stationary normal process with finite second spectral moment $\lambda_2 (= -r''(0))$ then the mean number of upcrossings of any fixed level u per unit time is finite and given by

$$(6.5) \qquad E(N_u(1)) = \frac{\sqrt{\lambda_2}}{2\pi} e^{-u^2/2}.$$

(Hence also all upcrossings are strict.)

PROOF This follows from the case u fixed, in the above lemma, together with (iii) of Lemma 6.3. □

The above discussion has been in terms of upcrossings. Clearly, similar results hold for downcrossings. In particular, the mean number of downcrossings is also given by (6.5).

In discussing the maximum of a stationary normal process $\xi(t)$ we shall find it useful to compare ξ with a very simple normal process $\xi^*(t)$ whose maximum is easily calculated using properties of its upcorssings. Specifically let η, ζ be independent standard normal r.v.'s and define

$$(6.6) \qquad \xi^*(t) = \eta \cos \omega t + \zeta \sin \omega t$$

where ω is a fixed positive constant.

It is clear that $\xi^*(t)$ is normal and that $\xi^*(t), \ldots, \xi^*(t_n)$ are jointly normal for any choice of t_i. (This follows most simply from the observation that $\sum_1^n c_i \xi^*(t_i)$ is normal for any choice of t_i and c_i.) Thus $\xi^*(t)$ is a normal process and $E(\xi^*(t)) = 0$, Its covariance function is calculated at once to be

$$\begin{aligned}(6.7) \qquad r(\tau) &= E\{(\eta \cos \omega t + \zeta \sin \omega t)(\eta \cos \omega(t+\tau) + \zeta \sin \omega(t+\tau))\} \\ &= \cos \omega t \cos \omega(t+\tau) + \sin \omega t \sin \omega(t+\tau) \\ &= \cos \omega \tau.\end{aligned}$$

Thus $\xi^*(t)$ is weakly stationary and hence strictly so, being normal.

Write now $\eta = A \cos \phi$ and $\zeta = A \sin \phi$, with $0 \le \phi < 2\pi$. Then

(6.8) $\quad \xi^*(t) = A\cos(\omega t - \phi)$.

The Jacobian $\frac{\partial(\eta,\zeta)}{\partial(A,\phi)} = A$, and it follows simply that A, ϕ have joint density

$$f_{A,\phi}(x,y) = \frac{1}{2\pi} x e^{-x^2/2}, \quad x > 0, \; 0 \le y < 2\pi,$$

showing that A, ϕ are independent, A having the Rayleigh distribution $x e^{-x^2/2}$ $(x \ge 0)$ and ϕ being uniform over $[0,2\pi)$. The sample paths of ξ^* are thus cosine functions with angular frequency ω, and having independent random amplitude A and phase ϕ.

The distribution of the maximum $M^*(T)$ for this process can be obtained geometrically. However, it is more instructive (and simpler) to use properties of upcrossings. It is clear that $\lambda_2 = \omega^2$ for this process and, writing $N = N_u^*(T)$ for the number of upcrossings of u in $(0,T)$, we have

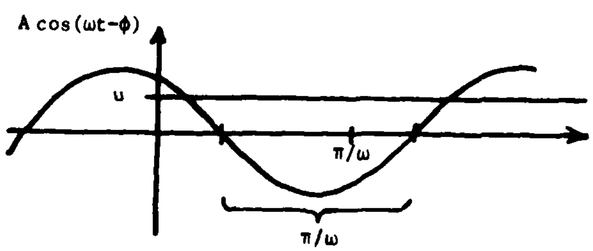
(6.9) $\quad E(N) = \frac{\omega T}{2\pi} e^{-u^2/2}$

and

$$P\{M^*(T) > u\} = P\{\xi^*(0) > u\} + P\{\xi^*(0) \le u, N \ge 1\}.$$

Now take $\omega T < \pi$. Then if $\xi^*(0) > u > 0$, the first upcrossing of u occurs after $t = \pi/\omega$ (see diagram), and hence $\{N \ge 1, \xi^*(0) > u\}$ is empty, so that

$$P\{\xi^*(0) \le u, N \ge 1\} = P\{N \ge 1\}.$$



Thus, since $N = 0$ or 1,

(6.10) $\quad P\{M^*(T) > u\} = 1 - \Phi(u) + P\{N \ge 1\} = 1 - \Phi(u) + E(N)$

$$= 1 - \Phi(u) + \frac{\omega T}{2\pi} e^{-u^2/2},$$

or equivalently,

(6.11) $P\{M^*(T) \le u\} = \Phi(u) - \frac{\lambda T}{2\pi} e^{-u^2/2}.$

As a matter of interest and for later use, it follows for this process that for fixed h, [illegible],

(6.12) $\frac{P\{M^*(h) > u\}}{h\phi(u)} \to \left(\frac{\lambda_2}{2\pi}\right)^{1/2}$ as $u \to \infty$

(since $1 - \Phi(u) \sim \phi(u)/u$ and $\lambda_2 = \lambda^2$). This limit in fact holds under much more general conditions, as we shall see.

As noted above we will want in the next chapter to compare a general stationary normal process with this special process. This comparison will be made by an application of the following easy consequence of Lemma 3.3.

<u>LEMMA 6.8</u> (Slepian) Let $\{\xi_1(t)\}$ and $\{\xi_2(t)\}$ be normal processes (possessing continuous sample functions but not necessarily being stationary). Suppose that these are standardized so that $E(\xi_1(t)) = E(\xi_2(t)) = 0$, $E(\xi_1^2(t)) = E(\xi_2^2(t)) = 1$, and write $r_1(t, s)$ and $r_2(t, s)$ for their covariance functions. Suppose that for some $\delta > 0$ we have $r_1(t, s) \le r_2(t, s)$ when $0 \le t, s \le \delta$. Then the respective maxima $M_1(t)$ and $M_2(t)$ satisfy

$$P\{M_1(T) \le u\} \le P\{M_2(T) \le u\}$$

when $0 \le T \le \delta$.

<u>PROOF</u> Define $M_n^{(1)}$ and $M_n^{(2)}$ relative to $\xi_1(t)$, $\xi_2(t)$ as in Lemma 6.1 where $\varepsilon_n = 2^{-n}$. Then, with probability one $M_n^{(1)} \to M_1(T)$, so that $\{M_n^{(1)} \le u\} \downarrow \{M_1(T) \le u\}$ and hence $P\{M_n^{(1)} \le u\} \to P\{M_1(T) \le u\}$ as $n \to \infty$. Similarly $P\{M_n^{(2)} \le u\} \to P\{M_2(T) \le u\}$. But it is clear from (3.6) of Lemma 3.2 that $P\{M_n^{(2)} \le u\} \le P\{M_n^{(1)} \le u\}$ so that the desired result follows. □

<u>Marked crossings</u>

The material in the remainder of this chapter will not be used until Chapter 8 and in subsequent chapters. The reader who wishes to do so may proceed directly to Chapter 7, and return to this section when needed.

We shall consider situations where we not only register the occurrence of an upcrossing, but also the value of some other random variable connected with the upcrossing. We may, e.g. be interested in the derivative $\xi'(t_i)$ at upcrossing points t_i of u by $\xi(\cdot)$ or the value $\xi(s_i)$ at downcrossing points s_i of zero by $\xi'(\cdot)$, i.e. at points where $\xi(t)$ has a local maximum. We shall refer to these as *marked crossings*, and for example regard $\xi'(t_i)$ and $\xi(s_i)$ as *marks* attached to the crossings at t_i and s_i. We shall here develop some methods for dealing with such marks, along similar lines to those leading to Rice's formula (although with some increase in complexity).

Let $\{\xi(t);\ t \geq 0\}$ and $\{\eta(t);\ t \geq 0\}$ be jointly stationary processes with continuous sample paths. Denote by t_i the upcrossings of u by $\xi(t)$, and let, for any interval A, $N_u(I;\ A)$ be the number of t_i in I such that $\eta(t_i) \in A$, and write $N_u(t;\ A) = N_u((0,\ t);\ A)$. $N_u(I)$, $N_u(t)$ will have the same meaning as before, e.g. $N_u(I) = N_u(I;\ (-\infty,\ \infty))$. Further define

$$J_q(u;\ A) = \frac{1}{q} P\{\xi(0) < u < \xi(q),\ \eta(0) \in A\}.$$

<u>LEMMA 6.9</u> Suppose $E(N_u(0,1)) < \infty$, let I be a bounded interval, let $q_n \to 0$ as $n \to \infty$, and let $N_n(A)$ be the number of points $jq_n \in I$ (with $(j-1)q_n \in I$) such that

$$\xi((j-1)q_n) < u < \xi(jq_n) \quad \text{and} \quad \eta((j-1)q_n) \in A.$$

Then

(i) if A is an open interval,

$$\liminf_{n\to\infty} N_n(A) \geq N_u(I;\ A), \quad \text{a.s.}$$

(ii) if, for every v,

(6.13) $P\{\xi(t) = u,\ \eta(t) = v \text{ for some } t \in I\} = 0$

then, for any interval A,

$$\limsup_{n\to\infty} N_n(A) \leq N_u(I;\ A), \quad \text{a.s.},$$

and

$$N_u(I;\ A) = \lim_{n\to\infty} N_n(A), \quad \text{a.s.},$$

(iii) if A is an open interval,

$$E(N_u(I;\ A)) \leq \liminf_{n\to\infty} E(N_n(A))$$

and, if (6.13) holds,

$$E(N_n(A)) \to E(N_u(I;\ A))$$

and

$$E(N_u(I;\ A)) = \lim_{q\downarrow 0} J_q(u;\ A).$$

<u>PROOF</u> (i) Suppose that $N_u(I;\ A) \geq m$ and that $\zeta(t)$ has upcrossings of u at $t_1, \ldots, t_m$ in the interior of I, with $\eta(t_i) \in A$, $i = 1, \ldots, m$. (Be the continuity of the distribution of $\zeta(t)$ no upcrossings occur at the endpoints of I.) Since $\eta(t)$ is continuous we can surround the t_i's by disjoint subintervals $(t_i - \varepsilon,\ t_i + \varepsilon)$ of I in which $\eta(t) \in A$. It then follows as in the proof of Lemma 6.3 (ii) that $\liminf_{n\to\infty} N_n(A) \geq m$.

(ii) First assume $N_u(I;\ A) = m < \infty$, and let $t_1, \ldots, t_m$ be as in (i). If (a,b) is the interior of A, (6.13) precludes $\eta(t_i) = a$ or b, so that $\eta(t_i) \in (a,b)$, and we may therefore take disjoint intervals $(t_i - \varepsilon,\ t_i + \varepsilon)$ in which $\eta(t) \in (a,b)$. Write J_n for the set of j's such that $(j-1)q_n$ and jq_n both belong to $(t_i - \varepsilon,\ t_i + \varepsilon)$ for some i, and J_n^* for the set of j's such that $(j-1)q_n$ and jq_n belong to I but $j \notin J_n$. Clearly t_i is the only upcrossing of u by $\zeta(t)$ for $t \in (t_i - \varepsilon,\ t_i + \varepsilon)$, and therefore by Lemma 6.2 (i),

$$\text{(6.14)} \quad \limsup_{n\to\infty} \sum_{j\in J_n} \chi_j \leq m,$$

where

$$\chi_j = \begin{cases} 1 & \text{if } \zeta((j-1)q_n) < u < \zeta(jq_n) \text{ and } \eta((j-1)q_n) \in A \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, if

$$\limsup_{n\to\infty} \sum_{j\in J_n^*} \chi_j > 0,$$

then for n arbitrarily large there are $j_n \in J_n^*$ with $\chi_{j_n} = 1$ and hence a sequence of integers $\{\tilde{n}\}$ such that $j_{\tilde{n}}q_{\tilde{n}} \to \tau$, with $\tau \notin (t_i - \varepsilon, t_i + \varepsilon)$, $i = 1, \ldots, m$, and $\chi_{j_{\tilde{n}}} = 1$. From the continuity of $\eta(t)$ it follows that $\eta(\tau) \in [a, b]$, and, since (6.13) precludes $\eta(\tau) = a$ or b, we must have $\eta(\tau) \in (a, b)$. Hence $\eta(t) \in (a, b) \subset A$ for $t \in (\tau - \varepsilon', \tau + \varepsilon')$ for some $\varepsilon' > 0$, which can be taken small enough to make $t_i \notin (\tau - \varepsilon', \tau + \varepsilon')$, $i = 1, \ldots, m$. Further, for $\tilde{n}$ large enough, both $(j_{\tilde{n}} - 1)q_{\tilde{n}}$ and $j_{\tilde{n}}q_{\tilde{n}}$ belong to $(\tau - \varepsilon', \tau + \varepsilon')$ and thus, by Lemma 6.2 (i), $\zeta(t)$ has a u-upcrossing in $(\tau - \varepsilon', \tau + \varepsilon')$ which contradicts $N_u(I; A) = m$. This shows that

$$\limsup_{n\to\infty} \sum_{j\in J_n^*} \chi_j = 0,$$

which together with (6.14) proves that $\limsup_{n\to\infty} N_n(A) \le N_u(I; A)$, a.s. Since furthermore, (6.13) implies that $N_u(I; A) = N_u(I;(a,b))$, part (i) gives that

$$\liminf_{n\to\infty} N_n(A) \ge \liminf_{n\to\infty} N_n((a,b)) \ge N_u(I;(a,b)) = N_u(I; A).$$

Hence $N_u(A) \to N_u(I; A) = m < \infty$ a.s. as asserted. If $N_u(I; A) = \infty$, the conclusion follows from part (i) with (a,b) replacing A, since $N_u(I; A) = N_u(I,(a,b))$ by (6.13).

(iii) The first conclusion follows at once form Fatou's Lemma and part (i), while it follows from part (ii) that $E(N_u(I; A)) = \lim_{n\to\infty} E(N_n(A))$, since $N_n(A) \le N_u(I)$ and $E(N_u(I)) < \infty$ by assumption. Further, if $I = (0,1)$, there are approximately q_n^{-1} points $jq_n \in I$, so that

$$E(N_n(A)) \sim q_n^{-1} E(\chi_1) = J_{q_n}(u; A).$$

Hence the last assertion of (iii) follows since the sequence $\{q_n\}$ is arbitrary. □

We shall now evaluate the limit of $J_q(u; A)$ for the case when $\zeta(t)$ and $\eta(t)$ are jointly normal processes. For convenience we assume that $\zeta(t)$ is standardized, i.e. it has mean zero and variance one. As was noted earlier, if $\zeta(t)$ is quadratic mean differentiable then $E(\zeta'(t)) = 0$, $\lambda_2 = \mathrm{Var}(\zeta'(t)) < \infty$, and $\zeta(t)$, $\zeta'(t)$ are independent for each t and normal, and hence they have the joint density function

$$p(u,\ z) = \phi(u)\lambda_2^{-1/2}\phi(z\lambda_2^{-1/2}).$$

Further it can be shown that the three processes $\{\zeta(t)\}$, $\{\zeta'(t)\}$, and $\{\eta(t)\}$ are jointly normal, and that the crosscovariances and covariances can be obtained as limits, e.g.

$$\mathrm{Cov}(\zeta'(t),\ \eta(t+\tau)) = \lim_{h\to 0} E(h^{-1}(\zeta(t+h) - \zeta(t)),\ \eta(t+\tau)).$$

Conditional distributions can also be defined, using ratios of density functions when they exist, e.g. for a measurable set A, we define

$$P\{\eta(0) \in A \mid \zeta(0) = u,\ \zeta'(0) = z\}$$

$$= \int_{y\in A} p_{\zeta(0),\zeta'(0),\eta(0)}(u,z,y)/p(u,z)\, dy,$$

where $p_{\zeta(0),\zeta'(0),\eta(0)}$ is the density function of $\zeta(0)$, $\zeta'(0)$, $\eta(0)$. In the sequel, conditional probabilities will always be understood as defined in this way.

<u>LEMMA 6.10</u> Let $\{\zeta(t)\}$ be a zero mean normal process, jointly normal with the process $\{\eta(t)\}$, and such that $\zeta(0)$, $\zeta'(0)$, $\eta(0)$ have a non-singular distribution. Assume further that $\{\eta(t)\}$ has continuous sample paths and that $\zeta(t)$ is differentiable in quadratic mean with $\lambda_2 = \mathrm{Var}(\zeta'(t))(< \infty)$. Then, for any measurable set A and any u,

$$\lim_{q\downarrow 0} J_q(u;\ A) = \int_{z=0}^{\infty} z p(u,z) P\{\eta(0) \in A \mid \zeta(0) = u,\ \zeta'(0) = z\} dz.$$

<u>PROOF</u> Write $\eta = \eta(0)$, and as in the proof of Lemma 6.6 introduce the independent normal r.v.'s $\zeta_1 = (\zeta(0) + \zeta(q))/2$, $\zeta_2 = (\zeta(q) - \zeta(0))/q$ with

variances $\sigma_1^2 = (r(0) + r(q))/2$, $\sigma_2^2 = 2(r(0) - r(q))/q^2$, and note that

$$J_q(u;\ A) = q^{-1}P\{|\zeta_1 - u| < \frac{q\zeta_2}{2},\ \eta \in A\}$$

$$= (q\sigma_1\sigma_2)^{-1}\int_{z=0}^{\infty}\int_{x=u-qz/2}^{u+qz/2}\phi\left(\frac{x}{\sigma_1}\right)\phi\left(\frac{z}{\sigma_2}\right)P\{\eta \in A|\zeta_1 = x,\ \zeta_2 = z\}dxdz$$

(6.15) $$= \int_{z=0}^{\infty}\frac{z}{\sigma_2}\phi\left(\frac{z}{\sigma_2}\right)\int_{x=-1}^{1}\frac{1}{2\sigma_1}\phi\left(\frac{u+xqz/2}{\sigma_1}\right)P\{\eta \in A|\zeta_1 = u+xqz/2,\ \zeta_2 = z\}dxdz$$

To obtain the limit of the conditional normal probability $P\{\eta \in A|\zeta_1 = v,\ \zeta_2 = z\}$ as $q \to 0$ (and $v \to u$) we note that since $\{\zeta(t)\}$ and $\{\eta(t)\}$ are jointly normal processes, the conditional distribution of $\eta = \eta(0)$ given $\zeta_1 = (\zeta(0) + \zeta(q))/2 = v$, $\zeta_2 = (\zeta(q) - \zeta(0))/q = z$, is also normal with mean

$$m_q(v,z) = E(\eta) + v\sigma_1^{-2}\ \mathrm{Cov}(\eta,\zeta_1) + z\sigma_2^{-2}\ \mathrm{Cov}(\eta,\zeta_2)$$

and variance

$$V_q = \mathrm{Var}(\eta) - \sigma_1^{-2}\ \mathrm{Cov}^2(\eta,\zeta_1) - \sigma_2^{-2}\ \mathrm{Cov}^2(\eta,\zeta_2).$$

Since $\zeta_1 \to \zeta(0)$, $\zeta_2 \to \zeta'(0)$ in quadratic mean as $q \to 0$ it follows that $\mathrm{Cov}(\eta,\zeta_1) \to \mathrm{Cov}(\eta(0),\zeta(0))$, $\mathrm{Cov}(\eta,\zeta_2) \to \mathrm{Cov}(\eta(0),\zeta'(0))$ as $q \to 0$. Since furthermore, $\sigma_1^2 \to r(0) = \mathrm{Var}(\zeta(0))$, $\sigma_2^2 \to \lambda_2 = \mathrm{Var}(\zeta'(0))$, and $\zeta(0)$, $\zeta'(0)$, $\eta(0)$ are non-singular by assumption we have $V_0 = \lim_{q\to 0} V_q > 0$. Thus, with $m_0 = \lim_{q\to 0} m_q(u + xqz/2, z)$, dominated convergence gives that for all x and z,

$$P\{\eta \in A|\zeta_1 = u+xqz/2,\ \zeta_2 = z\} = \int_A \frac{1}{\sqrt{V_q}}\phi\left(\frac{y - m_q}{\sqrt{V_q}}\right)dy$$

$$\to \int_A \frac{1}{\sqrt{V_0}}\phi\left(\frac{y - m_0}{\sqrt{V_0}}\right)dy$$

$$= P\{\eta(0) \in A|\zeta(0) = u,\ \zeta'(0) = z\}$$

as $q \to 0$. Again by dominated convergence it follows that

$$J_q(u;\ A) \to \int_{z=0}^{\infty}\frac{z}{\sqrt{\lambda_2}}\phi\left(\frac{z}{\sqrt{\lambda_2}}\right)\frac{1}{\sqrt{r(0)}}\phi\left(\frac{u}{\sqrt{r(0)}}\right)P\{\eta(0) \in A|\zeta(0) = u,\ \zeta'(0) = z\}dz$$

which is the conclusion of the lemma. □

Local maxima

As an application of the marked crossings theory we end this chapter with some comments concerning *local maxima*. To avoid technicalities we assume that the stationary normal process $\{\xi(t)\}$ has sample functions which are, with probability one, everywhere continuously differentiable. Sufficient conditions for this can be found in Cramér and Leadbetter (1967), and they require slightly more than finiteness of the second spectral moment λ_2; cf. the condition (6.3) for sample function continuity.

Clearly then $\xi(t)$ has a local maximum at t_0 if and only if $\xi'(t)$ has a downcrossing of zero at t_0, and a number of results for local maxima can therefore trivially be obtained from corresponding results for downcrossings.

In particular, to ensure that $\xi(t)$ has only finitely many local maxima in a finite time, we need the assumption $\lambda_4 = r^{(4)}(0) < \infty$, where λ_4 is the fourth spectral moment $\int_{-\infty}^{\infty} \lambda^4 \, dF(\lambda)$.

If $\lambda_4 < \infty$, then $\xi(t)$ has also a second derivative $\xi''(t)$, defined in quadratic mean, and $\xi(t)$, $\xi'(t)$, $\xi''(t)$ are jointly normal with mean zero and the covariance matrix

$$\begin{bmatrix} \lambda_0 & 0 & -\lambda_2 \\ 0 & \lambda_2 & 0 \\ -\lambda_2 & 0 & \lambda_4 \end{bmatrix},$$

where we usually assume $\lambda_0 = 1$. Further $\xi(t)$, $\xi'(t)$, $\xi''(t)$ have a non-singular distribution provided ξ is not of the form $\xi(t) = A\cos(\omega t - \phi)$. (In fact, the determinant of the covariance matrix is $\lambda_2(\lambda_0\lambda_4 - \lambda_2^2) = \lambda_2\{\int dF(\lambda) \int \lambda^4 \, dF(\lambda) - (\int \lambda^2 \, dF(\lambda))^2\}$, which is zero only if F is concentrated at two symmetric points.) If $\lambda_4 < \infty$ we also have the analogue of (6.2),

$$\mathrm{Cov}(\xi'(t), \xi'(t+\tau)) = -r''(\tau) = \lambda_2 - \frac{1}{2}\lambda_4\tau^2 + o(\tau^2) \quad \text{as } \tau \to 0,$$

and, normalizing to variance one, we obtain

$$\mathrm{Cov}(\lambda_2^{-1/2}\xi'(t), \lambda_2^{-1/2}\xi'(t+\tau)) = 1 - \frac{1}{2}\frac{\lambda_4}{\lambda_2}\tau^2 + o(\tau^2) \quad \text{as } \tau \to 0. \tag{6.16}$$

We will temporarily use the notation $N'(T)$ for the number of local maxima of $\xi(t)$, $0 < t < T$. From Rice's formula (6.5) and (6.16) we obtain that the expected number of local maxima in $(0,T)$ is

$$E(N'(T)) = \frac{T}{2\pi}\left(\frac{\lambda_4}{\lambda_2}\right)^{1/2},$$

In Chapter 9 we shall study heights and locations of *high local maxima*. Write $N_u'(T)$ for the number of local maxima of $\xi(t)$, $0 < t < T$, whose height exceeds u, i.e. with the previous notation, if $\xi(t)$ has local maxima at the time points $\{s_i\}$, then $N_u'(T)$ is the number of $s_i \in (0,T)$ such that $\xi(s_i) > u$.

<u>LEMMA 6.11</u> If $\{\xi(t)\}$ is stationary normal, with continuously differentiable sample paths, and with a quadratic mean second derivative $\xi''(t)$ with $\mathrm{Var}(\xi''(t)) = \lambda_4 < \infty$ such that $\xi(t)$, $\xi'(t)$, $\xi''(t)$ have a non-singular distribution, then

$$E(N_u'(T)) = T \int_{x=u}^{\infty} \int_{z=-\infty}^{0} |z|\, p(x,0,z)\, dz\, dx,$$

where $p(x,y,z)$ is the joint density of $\xi(t)$, $\xi'(t)$, $\xi''(t)$.

<u>PROOF</u> We shall use Lemmas 6.9 and 6.10, identifying $\zeta(t) = \xi(t)$ and $\eta(t) = \xi'(t)$. By assumption, $\{\eta(t)\}$ and $\{\zeta(t)\}$ satisfy the hypotheses of Lemma 6.10, with $\mathrm{Var}(\eta'(t)) = \lambda_4$, so that for any open interval A,

$$(6.17)\quad \lim_{q \downarrow 0} J_q(0;\, A) = \int_{z=0}^{\infty} z f_{\eta(0),\eta'(0)}(0,z)\, P\{\zeta(0) \in A \mid \eta(0) = 0,\ \eta'(0) = z\}\, dz =$$

$$= \int_{z=-\infty}^{0} |z|\, p(0,z)\, P\{\xi(0) \in A \mid \xi'(0) = 0,\ \xi''(0) = z\}\, dz,$$

where $p(0,z)$ is the density of $\xi'(0)$, $\xi''(0)$. By Theorems 6.5 and 6.7, all t such that $\zeta(t) = \xi'(t) = 0$, are either upcrossing or downcrossing points. Lemma 6.9 (iii) implies that, writing $N_0(T;\, V_\varepsilon)$ for the number of maxima in $(0,T)$ with height in $V_\varepsilon = (v - \varepsilon,\ v + \varepsilon)$,

$$P\{\xi'(t) = 0,\ \xi(t) = v \text{ for some } t \in (0,T)\} \le 2\,E(N_0(T;\, V_\varepsilon))$$

$$\le 2\,T \liminf_{q \downarrow 0} J_q(0;\, V_\varepsilon),$$

since $E(N_n(V_\varepsilon)) \sim J_q(0; V_\varepsilon)$. By (6.17) the right hand side can be made arbitrarily small. Thus $\xi(t)$, $\eta(t)$ satisfy condition (6.13) and by Lemma 6.9(iii) and stationarity

$$E(N_0(T;(u,\infty))) = T\, E(N_0(1;(u,\infty))) = T \lim_{q\to 0} J_q(0;(u,\infty)).$$

Inserting

$$P\{\xi(0) \in (u,\infty) \mid \xi'(0) = 0,\ \xi''(0) = z\} = \int_u^\infty p(x,0,z)/p(0,z)\,dx$$

into (6.17), the lemma follows. □

By inserting the normal density

$$p(x,0,z) = (2\pi)^{-3/2}(\lambda_2 D)^{-1/2}\exp(-(\lambda_4 x^2 + 2\lambda_2 xz + z^2)/2D),$$

where $D = \lambda_4 - \lambda_2^2$, we obtain after some calculation

(6.18) $$E(N_u'(T)) = \frac{T}{2\pi}\left\{\left(\frac{\lambda_4}{\lambda_2}\right)^{1/2}\left(1 - \Phi(u(\lambda_4/D)^{1/2})\right) + \right.$$

$$\left. + (2\pi\lambda_2)^{1/2}\,\phi(u)\,\Phi(u\lambda_2/D^{1/2})\right\}$$

CHAPTER 7

EXTREMAL THEORY OF MEAN SQUARE DIFFERENTIABLE NORMAL PROCESSES

In this chapter, the extremal theory of stationary normal processes will be developed – giving analogous results to those of Chapter 3. We shall assume throughout this chapter that $\{\xi(t); t \geq 0\}$ is a stationary, normal process with $E(\xi(t)) = 0$, $E(\xi^2(t)) = 1$, $E(\xi(t)\xi(t+\tau)) = r(\tau)$ where the spectral moment $\lambda_2 = r''(0)$ exists finite. Equivalently this requires that the mean number of upcrossings of any level per time unit is finite (Theorem 6.7), and also equivalently that the covariance function has the following representation,

(7.1) $\quad r(\tau) = 1 - \lambda_2 \tau^2/2 + o(\tau^2)$ as $\tau \to 0$.

Less regular cases where $\lambda_2 = \infty$ will be considered in Chapter 12.

As for normal sequences, the double exponential limit

$$P\{a_T(M(T) - b_T) \leq x\} \to \exp(-e^{-x}) \quad \text{as} \quad T \to \infty$$

(for $M(T) = \sup\{\xi(t); 0 \leq t \leq T\}$ as in Chapter 6) will be derived under the weak condition

(7.2) $\quad r(t) \log t \to 0$ as $t \to \infty$.

This is the continuous time analogue of (3.1), and it will be used to derive a version of Lemma 3.1, before starting the main development. Still weaker conditions corresponding to (3.12) will be obtained at the end of this chapter. In the following lemma we shall consider a level u which increases with the time period T in such a way that $E(N_u(T))$ remains constant, i.e. $T\mu$ remains constant, where $\mu = E(N_u(1)) = \frac{1}{2\pi} \lambda_2^{1/2} e^{-u^2/2}$.

We shall also consider points $\{kq; k = 1,2\ldots\}$ where q depends on u (or equivalently on T) and $q \to 0$ as u (or T) $\to \infty$. The statement that "a property holds provided $\Psi = \Psi(u) \downarrow 0$ sufficiently slowly" is to be taken to have the obvious meaning that there exists some $\Psi_0(u) \downarrow 0$ for which the property holds, and it holds for any

$\gamma(u)$ such that $\gamma(u) \to 0$ but $\gamma_0(u) \leq \gamma(u)$ as $u \to \infty$. The following is the promised continuous analogue of Lemma 3.1.

<u>LEMMA 7.1</u> Let $\varepsilon > 0$ be given.

(i) If (7.2) holds, then $\sup\{|r(t)|; |t| \geq \varepsilon\} = \delta < 1$.

(ii) Suppose that (7.1) and (7.2) both hold. Let $T \sim \tau/\mu$, where τ is fixed and $\mu = E(N_u(1)) = \frac{1}{2\pi}\lambda_2^{1/2} e^{-u^2/2}$, so that $u \sim (2\log T)^{1/2}$ as $T \to \infty$ (as is easily checked). If $qu = q(u)u \downarrow 0$ sufficiently slowly as $u \to \infty$ then

$$\frac{T}{q} \sum_{\varepsilon \leq kq \leq T} |r(kq)| e^{-u^2/(1+|r(kq)|)} \to 0 \quad \text{as} \quad T \to \infty.$$

<u>PROOF</u> (i) As in the discrete case (cf. remarks preceding Lemma 3.1) if $r(t) = 1$ for any $t > 0$, then $r(t) = 1$ for arbitrarily large values of t which contradicts (7.2). Hence $|r(t)| < 1$ for $|t| \geq \varepsilon$, and since $r(t)$ is continuous and tends to zero as $t \to \infty$, we must have $|r(t)|$ bounded away from 1 in $|t| \geq \varepsilon$, and (i) follows.

(ii) As in the discrete case, choose a constant β such that $0 < \beta < \frac{1-\delta}{1+\delta}$. Letting K be a generic constant,

$$\begin{aligned}\frac{T}{q} \sum_{\varepsilon \leq kq \leq T^\beta} |r(kq)| e^{-u^2/(1+|r(kq)|)} &\leq \frac{T^{\beta+1}}{q^2} e^{-u^2/(1+\delta)} \\ &= K\frac{T^{\beta+1}}{q^2} \mu^{2/(1+\delta)} \\ &\leq \frac{K}{q^2} T^{\beta+1-2/(1+\delta)} \\ &\leq \frac{K}{q^2u^2}(\log T)\, T^{\beta+1-2/(1+\delta)}\end{aligned}$$

since $u^2 \sim 2\log T$, as noted. If γ is chosen so that $0 < \gamma < \frac{1-\delta}{1+\delta} - \beta$, the last expression is dominated by $K(qu)^{-2}T^{-\gamma}$ which tends to zero provided $uq \to 0$ more slowly than $T^{-\gamma/2}$ $(= Ke^{-\gamma u^2/4})$. Hence this sum tends to zero.

By writing $e^{-u^2/(1+|r(kq)|)} = e^{-u^2} e^{u^2|r(kq)|/(1+|r(kq)|)}$ we see that the remaining sum does not exceed

$$\frac{T}{q} e^{-u^2} \sum_{T^\beta < kq \leq T} |r(kq)| e^{u^2|r(kq)|}.$$

Again as in the discrete case, if $\delta(t) = \sup_{s \geq t} |r(s) \log s|$ then $\delta(t) \to 0$ as $t \to \infty$ and for $s \geq t > 1$ we have $|r(s)| \leq \delta(t)/\log s \leq \delta(t)/\log t$. Thus for $kq \geq T^\beta$, $u^2|r(kq)| \leq K \log T\, \delta(T^\beta)/\log T^\beta = \frac{K}{\beta}\delta(T^\beta)$ which tends to zero, uniformly in k. Hence the exponential term $e^{u^2|r(kq)|}$ is certainly bounded in (k,u). It is thus sufficient to show that

$$\frac{T}{q} e^{-u^2} \sum_{T^\beta \leq kq \leq T} |r(kq)| \to 0 \quad \text{as} \quad T \to \infty.$$

But this does not exceed

$$K \frac{T}{q} T^{-2} \frac{T}{q}\, \delta(T^\beta)/\log T^\beta \leq K\, \delta(T^\beta)/(q^2u^2)$$

which again tends to zero provided $qu \to 0$ sufficiently slowly (i.e. slower than $\delta(T^\beta)^{1/2}$). □

Having proved this technical lemma, we now proceed to the main derivation of the extremal results under the assumption that $r''(0)$ exists (i.e. $\lambda_2 < \infty$) and (7.2) holds. The condition $\lambda_2 < \infty$ guarantees that the point process of upcrossings of a level u will have a finite intensity. The case $\lambda_2 = \infty$ is also of interest, and, as noted, will be treated in Chapter 12, but requires the use of more complex methods (such as an extended definition of upcrossings).

Our basic technique here is to divide the interval $(0,T)$ (where T becomes large) into n pieces of fixed length h $(n = [T/h])$. Then $M(T)$ will clearly be close to $M(nh)$ which is the maximum of n r.v.'s $\zeta_j = M((j-1)h, jh)$, $j = 1,2,\ldots,n$, (the $\{\zeta_j\}$ forming a stationary sequence). Thus we might expect that the methods used for sequences would apply here and this is the case (although we shall organize our arguments slightly differently to better suit the present purposes).

It is therefore not surprising that the tail of the distribution of the ξ_i, i.e. $P\{M(h) > u\}$ (for fixed h) plays a central role. In fact the same asymptotic form (6.12) holds for this tail probability here, as did for the special process $\xi^*(t) = \eta\cos\omega t + \zeta\sin\omega t$. In this present chapter it will be sufficient to obtain the following somewhat weaker result. In this we shall use Slepian's Lemma (Lemma 6.8) to compare maxima of $\xi(t)$ and $\xi^*(t)$ along the lines of a procedure originally used by S.M. Berman (1971 b).

<u>LEMMA 7.2</u> Suppose that the (standarized) stationary normal process $\{\xi(t)\}$ satisfies (7.1). Then, with the above notation,

(i) for all $h > 0$, $P\{M(h) > u\} \le 1 - \Phi(u) + \mu h$

so that $\limsup_{u\to\infty} P\{M(h) > u\}/(\mu h) \le 1$,

(ii) given $\theta < 1$ there exists $h_0 = h_0(\theta)$ such that for $0 \le h \le h_0$

(7.3) $$P\{M(h) > u\} \ge 1 - \Phi(u) + \theta\mu h$$

so that $\liminf_{u\to\infty} P\{M(h) > u\}/(\mu h) \ge \theta$ for $0 \le h \le h_0 = h_0(\theta)$.

<u>PROOF</u> (i) follows since

$$\begin{aligned} P\{M(h) > u\} &\le P\{\xi(0) > u\} + P\{N_u(h) \ge 1\} \\ &\le 1 - \Phi(u) + E(N_u(h)). \end{aligned}$$

The second result (ii) follows simply from Slepian's Lemma (Lemma 6.8) by comparison with the simple process $\xi^*(t)$ given by (6.6). For if $\omega = \theta\lambda_2^{1/2}$ we have, by (7.1), $r(t) \le \cos\omega t$ for $0 \le t \le h_0 < \pi/\omega$. ($h_0 = h_0(\theta) > 0$). But this shows that the covariance function of $\xi(t)$ is dominated by that of $\xi^*(t)$ in $[0, h_0]$ and hence $P\{M(h) > u\} \ge P\{M^*(h) > u\}$ for $h \le h_0$, (with M^* as in (6.10)), which then gives (ii). □

Our remaining task is to approximate the maximum $M(T)$ (for increasing T) by the maxima over suitable, separated, fixed length subintervals, and show asymptotic independence of the maxima over these intervals. First we give a simple but useful lemma. In this, for $q > 0$, N_u and $N_u^{(q)}$ will denote the number of upcrossings of u in a fixed interval I of length h, by the process $\{\xi(t)\}$, and the sequence $\{\xi(kq)\}$, respectively. More precisely, $N_u^{(q)}$ is the number of $kq \in I$ such that $(k-1)q \in I$ and $\xi((k-1)q) < u < \xi(kq)$ (cf. Lemma 6.3 with q for q_n and $N_n = N_u^{(q_n)}$).

<u>LEMMA 7.3</u> If (7.1) holds, with the above notation, as $u \to \infty$, $qu \to 0$,

(i) $E(N_u^{(q)}) = h\mu + o(\mu)$,

(ii) $P\{M(I) \le u\} = P\{\xi(kq) \le u,\ kq \in I\} + o(\mu)$,

where each $o(\mu)$-term is uniform in all such intervals I of length $h \le h_0$ for any fixed $h_0 > 0$.

<u>PROOF</u> The number of points $kq \in I$ with $(k-1)q \in I$ is clearly $(h/q) - \beta$ where $0 \le \beta \le 2$. Hence with $J_q(u)$ defined by (6.1), Lemma 6.6 implies that

$$\begin{aligned} E(N_u^{(q)}) &= (\frac{h}{q} + \beta)P\{\xi(0) < u < \xi(q)\} \\ &= (h + \beta q)J_q(u) \\ &= \mu h(1 + o(1)) + O(\mu q) \end{aligned}$$

where the o- and O-terms are uniform in h so that (i) clearly holds with $o(\mu)$ uniform in $0 < h < h_0$.

To prove (ii), note that if a is the left-hand endpoint of I,

$$\begin{aligned} 0 &\le P\{\xi(kq) \le u,\ kq \in I\} - P\{M(h) \le u\} \\ &\le P\{\xi(a) > u\} + P\{\xi(a) < u,\ N_u \ge 1,\ N_u^{(q)} = 0\} \\ &\le 1 - \Phi(u) + P\{N_u - N_u^{(q)} \ge 1\}. \end{aligned}$$

The first term is $o(\phi(u)) = o(\mu)$, independent of h. Since $N_u - N_u^{(q)}$ is a non-negative integer-valued random variable (cf. Lemma 6.3 (i)), the second term does not exceed $E(N_u - N_u^{(q)})$ which by (i) is $o(\mu)$, uniformly in $(0,h_0)$. Hence (ii) follows. □

Now let $u, T \to \infty$ in such a way that $T\mu \to \tau > 0$. Fix $h > 0$ and write $n = [T/h]$. Divide the interval $[0,nh]$ into n pieces each of length h. Fix ε, $0 < \varepsilon < h$ and divide each piece into two – of length $h - \varepsilon$ and ε, respectively. By doing so we obtain n pairs of intervals $I_1, I_1^*, \ldots, I_n, I_n^*$, alternately of length $h - \varepsilon$ and ε, making up the whole interval $[0,T]$ apart form one further piece which is contained in the next pair, I_{n+1}, I_{n+1}^*.

<u>LEMMA 7.4</u> As $u \to \infty$, $qu \to 0$ and $T\mu \to \tau > 0$,

(i) $\limsup_{T\to\infty} |P\{M(\bigcup_1^n I_j) \le u\} - P\{M(nh) \le u\}| \le \frac{\tau}{h}\varepsilon$

(ii) $P\{\xi(kq) \le u,\ kq \in \bigcup_1^n I_j\} - P\{M(\bigcup_1^n I_j) \le u\} \to 0.$

<u>PROOF</u> For (i) note that

$$\begin{aligned} 0 &\le P\{M(\bigcup_1^n I_j) \le u\} - P\{M(nh) \le u\} \\ &\le nP\{M(I_1^*) > u\} \\ &\sim \frac{\tau\varepsilon}{h} P\{M(I_1^*) > u\}/(\mu\varepsilon) \end{aligned}$$

since $n = [T/h] \sim \tau/(\mu h)$. Since I_1^* has length ε, (i) follows from Lemma 7.2 (i).

To prove (ii) we note that the expression on the left is non-negative and dominated by

$$\sum_{j=1}^{n} \left(P\{\xi(kq) \le u,\ kq \in I_j\} - P\{M(I_j) \le u\}\right)$$

which by Lemma 7.3 (ii) does not exceed $no(\mu) = [T/h]o(\mu) = o(1)$, (the $o(\mu)$-term being uniform in the I_j's), as required. □

The next lemma, implying the asymptotic independence of maxima, is formulated in terms of the condition (7.4), also appearing in Lemma 7.1.

<u>LEMMA 7.5</u> Suppose $r(t) \to 0$ as $t \to \infty$ and that, as $T \to \infty$, $qu \to 0$,

$$\frac{T}{q} \sum_{\varepsilon \le kq \le T} |r(kq)| e^{-u^2/(1+|r(kq)|)} \to 0 \tag{7.4}$$

for each $\varepsilon > 0$. Then as $T \to \infty$, $qu \to 0$, $T\mu \to \tau$,

(i) $\quad P\{\xi(kq) \leq u,\ kq \in \bigcup_1^n I_j\} - \prod_{j=1}^n P\{\xi(kq) \leq u,\ kq \in I_j\} \to 0$

(ii) $\quad \limsup_{T\to\infty} \left| \prod_{j=1}^n P\{\xi(kq) \leq u,\ kq \in I_j\} - P^n\{M(h) \leq u\} \right| \leq \frac{2\tau}{h}\,\varepsilon$

for each ε, $0 < \varepsilon < h$.

PROOF To show (i) we use Lemma 3.2, and compare the maximum of $\xi(kq)$, $kq \in \bigcup_1^n I_j$ under the full covariance structure, with the maximum of $\xi(kq)$, assuming variables arising from different I_j-intervals are independent. To formalize this, let $\Lambda^1 = (\lambda_{ij}^1)$ be the covariance matrix of $\xi(kq)$, $kq \in \cup I_j$ and let $\Lambda^0 = (\lambda_{ij}^0)$ be the modification obtained by writing zeros in the off-diagonal blocks (which would occur if the groups were independent of each other); e.g. with $n = 3$,

$$\Lambda^1 = \left[\begin{array}{c|c|c} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \hline \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ \hline \Lambda_{31} & \Lambda_{32} & \Lambda_{33} \end{array}\right], \quad \Lambda^0 = \left[\begin{array}{c|c|c} \Lambda_{11} & 0 & 0 \\ \hline 0 & \Lambda_{22} & 0 \\ \hline 0 & 0 & \Lambda_{33} \end{array}\right].$$

From Lemma 3.2 we obtain

$$(7.5) \qquad \left|P\{\xi(kq) \leq u,\ kq \in \bigcup_1^n I_j\} - \prod_{j=1}^n P\{\xi(kq) \leq u,\ kq \in I_j\}\right|$$
$$\leq \frac{1}{2\pi} \sum_{1\leq i<j\leq L} |\lambda_{ij}^1 - \lambda_{ij}^0| (1 - \rho_{ij}^2)^{-1/2} \exp(-u^2/(1 + \rho_{ij}))$$

where L is the total number of kq-points in $\bigcup_1^n I_j$, and $\rho_{ij} = |\lambda_{ij}^1|$. Since all terms with i,j in the same diagonal block vanish, while otherwise $\sup \rho_{ij} = \delta < 1$ by Lemma 7.1 (i), we see that the double sum does not exceed

$$K\, \Sigma^*\, \rho_{ij} \exp(-u^2/(1 + \rho_{ij})),$$

where Σ^* indicates that the summation is carried out over $i < j$ with (i,j) in the off-diagonal blocks only. But ρ_{ij} is of the form $|r(kq)|$ where there are not more than T/q terms with the same k-value. Thus, since the minimum value of kq is at least ε, we obtain the bound

$$P\{\xi(kq) \le u,\ kq \in \bigcup_1^n I_j\} - \prod_{j=1}^n P\{\xi(kq) \le u,\ kq \in I_j\}$$

$$\le K \frac{T}{q} \sum_{\varepsilon \le kq \le T} |r(kq)| e^{-u^2/(1+|r(kq)|)}$$

which tends to zero by Assumption (7.4) so that (i) follows.

To prove (ii), note that by Lemma 7.3 (ii),

$$0 \le P\{\xi(kq) \le u,\ kq \in I_j\} - P\{M(I_j) \le u\} = o(\mu)$$

(uniformly in j) and

$$0 \le P\{M(I_j) \le u\} - P\{M(h) \le u\} = P\{M(I_j^*) > u\}$$

so that by Lemma 7.2 (i), for sufficiently large n (uniformly in j)

$$0 \le P_j - P \le 2\mu\varepsilon$$

where $P_j = P\{\xi(kq) \le u,\ kq \in I_j\}$, $P = P\{M(h) \le u\}$. Hence

$$0 \le \prod_{j=1}^n P_j - P^n \le (\max P_j)^n - P^n \le 2n\mu\varepsilon$$

(using the fact that $y^n - x^n \le n(y-x)$ for $0 < x < y < 1$). Part (ii) now follows since $n\mu \sim T\mu/h \to \tau/h$. □

The basic extremal theorem now follows readily.

<u>THEOREM 7.6</u> Let $u, T \to \infty$ in such a way that $T\mu = \frac{T}{2\pi} \lambda_2^{1/2} e^{-u^2/2} \to \tau > 0$. Suppose that $r(t)$ satisfies (7.1) and either (7.2) or the weaker condition (7.4) (cf. Lemma 7.1). Then

(7.6) $\qquad P\{M(T) \le u\} \to e^{-\tau}$ as $T \to \infty$.

<u>PROOF</u> By Lemma 7.1 the assumption (7.4) of Lemma 7.5 holds. From Lemma 7.4 and 7.5 we obtain

$$\limsup_{T\to\infty} |P\{M(nh) \le u\} - P^n\{M(h) \le u\}| \le K\varepsilon$$

for some K independent of ε, and since $\varepsilon > 0$ is arbitrary it follows that

$$P\{M(nh) \le u\} - P^n\{M(h) \le u\} \to 0.$$

Further, since $nh \le T < (n+1)h$, it follows along now familiar lines that

$$0 \le P\{M(nh) \le u\} - P\{M(T) \le u\} \le P\{N_u(h) \ge 1\} \le \mu h$$

which tends to zero, so that

$$P\{M(T) \le u\} = P^n\{M(h) \le u\} + o(1).$$

This holds for any fixed $h > 0$. Suppose now that θ is fixed, $0 < \theta < 1$, and h chosen with $0 < h < h_0$ where $h_0 = h_0(\theta)$ is as in Lemma 7.2 (ii), from whence it follows that

$$P\{M(h) > u\} \ge \theta\mu h(1 + o(1)) = \frac{\theta\tau}{n}(1 + o(1))$$

and hence

$$P\{M(T) \le u\} = (1 - P\{M(h) > u\})^n + o(1)$$
$$\le (1 - \theta\tau/n + o(1/n))^n + o(1)$$

so that

$$\limsup_{T\to\infty} P\{M(T) \le u\} \le e^{-\theta\tau}$$

By letting $\theta \uparrow 1$ we see that $\limsup P\{M(T) \le u\} \le e^{-\tau}$. That the opposite inequality holds for the lim inf is seen in a similar way, but even more simply, from Lemma 7.2 (i) (no θ being involved) so that the entire result follows. □

COROLLARY 7.7 Suppose the conditions of the theorem hold, and let $E = E_T$ be any interval of length γT for a constant $\gamma > 0$. Then $P\{M(E) \le u\} \to e^{-\gamma\tau}$ as $T \to \infty$.

PROOF By stationarity we may take E to be an interval with left end-point at zero, so that $P\{M(E) \le u\} = P\{M(\gamma T) \le u\}$. It is simply checked that the process $\eta(t) = \xi(\gamma t)$ satisfies the conditions of the theorem, and has mean number of upcrossings per unit time given by $\mu_\eta = \gamma\mu$, so that $\mu_\eta T \to \gamma\tau$. Writing M_η for the maximum of η the result follows at once since $P\{M(\gamma T) \le u\} = P\{M_\eta(T) \le u\} \to e^{-\gamma\tau}$. □

It is now a simple matter to obtain the double exponential limiting law for $M(T)$ under a linear normalization. This is similar to the result of Theorem 3.5 for normal sequences.

<u>**THEOREM 7.8**</u> Suppose that the (standardized) stationary normal process $\{\xi(t)\}$ satisfies (7.1) and (7.2) (or (7.4)). Then

$$P\{a_T(M(T) - b_T) \le x\} \to \exp(-e^{-x}) \quad \text{as} \quad T \to \infty \tag{7.7}$$

where

$$\begin{aligned} a_T &= (2 \log T)^{1/2} \\ b_T &= (2 \log T)^{1/2} + (\log \frac{\lambda_2^{1/2}}{2\pi})/(2 \log T)^{1/2} . \end{aligned} \tag{7.8}$$

<u>PROOF</u> Write $\tau = e^{-x}$ and define

$$u^2 = 2(\log T + x + \log(\lambda_2^{1/2}/2\pi)) \tag{7.9}$$

so that

$$T\mu = T(\lambda_2^{1/2}/2\pi)\, e^{-u^2/2} = e^{-x} = \tau .$$

Hence (7.6) holds. But it follows from (7.9) that

$$\begin{aligned} u &= (2 \log T)^{1/2}\left[1 + \frac{x + \log(\lambda_2^{1/2}/2\pi)}{2 \log T} + o(\frac{1}{\log T})\right] \\ &= \frac{x}{a_T} + b_T + o(a_T^{-1}) \end{aligned}$$

so that (7.6) gives $P\{a_T(M(T) - b_T) + o(1) \le x\} \to e^{-\tau}$ from which (7.7) follows at once. □

It is of interest to note in passing that this calculation is somewhat simpler – due to the absence of a $\log u$-term in (7.9), than the corresponding calculation in the discrete case (cf. Theorem 1.11).

In the discrete case we obtained Poisson limiting behaviour for the exceedances of a high level. Corresponding results hold for the point processes of high level upcrossings under the conditions of this chapter. These are readily obtained from the present extremal theory by means of our familiar point process convergence theorem, as in the discrete case, resulting in a number of interesting consequences concernint local maxima, length of excursions, etc. We will defer such a discussion to Chapters 8 and 9. However, it is worth noting here that historically the asymptotic Poisson distribution of the number of high level upcrossings was proved first (under more restrictive conditions) by Volkonski and Rozanov (1961). Cramér (1965) noted the connection with the maximum given e.g. by

$$\{N_u(T)=0\} = \{M(T) \le u\} \cup \{N_u(T)=0,\ \xi(0)>u\},$$

which led to the determination of the asymptotic distribution of $M(T)$, and subsequent extremal development.

Extremal results under weaker conditions at infinity

As already noted, the above extremal results may be generalized by weakening of either (7.1) or (7.2). The weakening of the "local condition" (7.1) by allowing $\lambda_2 = \infty$ is somewhat more complicated and will be described in Chapter 12. For a weakening of (7.2) - describing the behaviour of the correlation at distant points - we may proceed by similar means to those used in the discrete case, and we devote the remainder of the present chapter to this, following Leadbetter, Lindgren and Rootzén (1979) and Mittal (1979). Of course we cannot expect a substantial weakening of (7.2) since it is clearly close to being a necessary condition.

Let $h(t)$ be any function and define

$$(7.10)\qquad \begin{aligned} \theta_T(h) &= \{t \in (0,T];\ |r(t)|\log t > h(t)\} \\ \ell_T(h) &= \lambda(\theta_T(h)) = \text{Lebesgue measure of } \theta_T(h). \end{aligned}$$

By analogy with the conditions for discrete time we will place restrictions on the amount of time that $|r(t)|\log t$ is large by requiring that there is some non-increasing function h with $h(t) \downarrow 0$ as $t \uparrow \infty$ such that

(7.11) $\ell_T(h) = O(T/(\log T)^{\gamma})$, for some $\gamma > 1/2$,

and some constant $K > 0$ such that

(7.12) $\ell_T(K) = O(T^{\eta})$, for some $\eta < 1$.

Obviously the condition $r(t)\log t \to 0$ as $t \to \infty$, implies that $\ell_T(h)$ is empty if e.g. $h(t) = \sup_{s\geq t} |r(s)| \log s$, so that (7.11) is actually weaker than (7.2). In fact, (7.11) is also weaker than some other conditions which have been used on occasions. For example, since $\int_0^T |r(t)|^p dt \geq \ell_T(h)(h(T)/\log T)^p$ if h is decreasing, $\int_0^\infty r^2(t)dt < \infty$ implies that $\ell_T(h) = O((\log T/h(T))^2)$ for all h, so that (7.11) is indeed weaker than the condition $\int_0^\infty r^2(t)dt < \infty$, sometimes used in the literature.

<u>THEOREM 7.9</u> Let $u = u_T \to \infty$ so that $T\mu \to \tau > 0$, and suppose $r(t) \to 0$ as $t \to \infty$, and furthermore satisfies (7.1), (7.11), and (7.12). Then

$$P\{M(T) \leq u\} \to e^{-\tau} \quad \text{as} \quad T \to \infty.$$

<u>PROOF</u> By Theorem 7.6 we have only to show that $q \to 0$ may be chosen so that $qu \to 0$, and (7.4) holds, i.e. for $\varepsilon > 0$,

$$(7.13) \quad \frac{T}{q} \sum_{\varepsilon\leq kq\leq T} |r(kq)| e^{-u^2/(1+|r(kq)|)} \to 0.$$

Let $\delta(t) = \sup_{s\geq t} |r(s)|$, let β satisfy $0 < \beta < (1-\delta(\varepsilon))/(1+\delta(\varepsilon))$, and split the sum in (7.13) into two parts at $kq \approx T^{\beta}$, i.e. let Σ' be the sum over $\varepsilon \leq kq \leq T^{\beta}$ and Σ'' the sum over $T^{\beta} < kq \leq T$. Since

$$e^{-u^2/2} = O(1)/T$$

we can estimate Σ' simply from the number of terms, as follows,

$$\frac{T}{q}\Sigma' = \frac{T}{q} \sum_{\varepsilon\leq kq\leq T^{\beta}} |r(kq)| e^{-u^2/(1+|r(kq)|)}$$

$$\leq \frac{T}{q} \cdot \frac{T^{\beta}}{q} \cdot e^{-u^2/(1+\delta(\varepsilon))} \leq \frac{K}{q^2} T^{1+\beta-2/(1+\delta(\varepsilon))}$$

$$= \frac{K}{q^2u^2} T^{1+\beta-2/(1+\delta(\varepsilon))} \log T \to 0$$

if $qu \to 0$ slowly enough.

For the remaining sum Σ'' we need a bound on the number of terms for which $|r(kq)| \log kq$ is not bounded by a small function. Define, for a function h,

$$n_T(h) = \#\{k;\ T^{\delta} < kq \le T,\ |r(kq)| \log kq > h(kq)\}$$

(where $\#$ denotes cardinality) in analogy with $\lambda_T(h)$ in (7.10). Since $\lambda_2 < \infty$ and therefore r has a bounded derivative,

$$|r(t+h) - r(t)| \le C|h|$$

for some constant C, and this can be used to give a bound for $n_T(h)$ in terms of $\lambda_T(h/2)$. In fact, we will see that

$$n_T(h) \le C'(\log T/h(T))\, \lambda_T(h/2), \tag{7.14}$$

if T is large enough. Since, for $t \ge kq$, $|r(t)| \log t \ge (|r(kq)| - C|t-kq|) \log kq$ we have that if

$$|r(kq)| \log kq > h(kq)$$

and t is such that

$$kq < t < kq + \frac{h(T)}{2C \log T}$$

then

$$|r(t)| \log t > h(t)/2.$$

Since $u \sim \sqrt{2 \log T}$, $h(T)/2C \log T \sim (h(T)/Cqu) \cdot q/u < q$ for large T if $qu \to 0$ slowly enough. This implies that for T large enough, the kq which contribute to $n_T(h)$ also contribute disjoint intervals of length at least $h(T)/2C \log T$ to $\lambda_T(h/2)$, and we get (7.14) with $C' = 2C$.

We can now proceed by splitting the sum Σ'' according to whether $kq \in \theta_T(2K)$ or not. Recalling the notation $\delta(t) = \sup_{s \ge t} |r(s)|$, we have

$$\frac{T}{q}\Sigma'' = \frac{T}{q}\sum_{T^{\beta}<kq\le T} r(kq)\, e^{-u^2/(1+r(kq))} \le \frac{T}{q} n_T(2K)\, e^{-u^2/(1+\delta(T^{\beta}))}$$

$$+ \frac{T}{q}\sum_{T^{\beta}<kq\le T,\ kq\in\theta_T(2K)^*} r(kq)\, e^{-u^2(1-2K/\log T^{\beta})}$$

(where the $*$ denotes complementation). The first term in (7.15) is bounded by

$$\frac{T}{q} C'(\log T/2K)\,\ell_T(K)\,O(1)\,T^{-2/(1+\delta(T^{\beta}))} \le \frac{C''}{qu}(\log T)^{3/2} T^{1+\eta-2/(1+\delta(T^{\beta}))}$$

Since $\eta < 1$ by (7.12) and $\delta(T^{\beta}) \to 0$, this bound tends to zero as $T \to \infty$ and $qu \to 0$ slowly enough.

The second term in (7.15) is bounded by

$$(7.16)\qquad \left(\frac{T}{q}\right)^2 e^{-u^2(1-2K/(\beta\log T))}\,\frac{1}{\beta\log T}\cdot\frac{q}{T}\,\Sigma|r(kq)|\log kq = F_1\cdot F_2,$$

say, where the sum is extended over all kq such that $T^{\beta} < kq \le T$ and $kq \in \theta_T(2K)^*$. We will see that F_1 may tend slowly to infinity, but $F_2 \to 0$ as $T \to \infty$ so that $F_1\cdot F_2 \to 0$. We start with F_2, introducing the function h that appears in (7.11) and split the sum according to whether $kq \in \theta_T(2h)$ or not, giving

$$\begin{aligned}
F_2 &= \frac{q}{T}\,\Sigma|r(kq)|\log kq \\
&\le \frac{q}{T}\sum_{\substack{kq\in\theta_T(2h)^*\\ kq\le T}} + \frac{q}{T}\sum_{kq\in\theta_T(2h)\cap\theta_T(2K)^*} \\
&\le \frac{q}{T}\cdot\frac{T}{q}\,2h(T^{\beta}) + \frac{q}{T}\cdot 2Kn_T(2h) \\
&\le 2h(T^{\beta}) + 2KC'\,\frac{q}{T}(\log T/h(T))\,\ell_T(h) \\
&= 2h(T^{\beta}) + h(T)^{-1}(\log T)^{1-1/2-\gamma}(qu)\cdot O(1) \\
&= 2h(T^{\beta}) + k(T)(qu),
\end{aligned}$$

say, by condition (7.11). Since $1/2 - \gamma < 0$, we can deduce that $k(T) \to 0$ as $T \to \infty$, provided $h(t)$ decreases sufficiently slowly. Note that if (7.11) is satisfied for some function h, then it is satisfied for all functions which decrease more slowly. We therefore assume that $k(T) \to 0$ as $T \to \infty$. The remaining factor F_1 in (7.16) is given by

$$F_1 = \left(\frac{T}{q}\right)^2 e^{-u^2(1-2K/(\beta \log T))} \frac{1}{\beta \log T} .$$

Using the fact that $u^2 = 2\log T + O(1)$ we obtain

$$F_1 = \frac{O(1)}{q^2 \log T} = O(1)/(qu)^2 .$$

Thus, for some $C > 0$,

$$F_1 \cdot F_2 \le C\left\{\frac{2h(T^\beta)}{q^2u^2} + \frac{k(T)}{qu}\right\} .$$

Since $k(T)$ does not depend on the choice of q we may choose $qu \to 0$ sufficiently slowly so that both terms tend to zero, which completes the proof of the theorem. □

REMARK 7.10 As in discrete time, one would be inclined to consider a condition like

$$(7.17) \quad \frac{q}{T} \sum_{T^\beta \le kq \le T} |r(kq)| \log kq \; e^{\gamma |r(kq)| \log kq} \to 0$$

as $T \to \infty$, for some $\beta < 1$, $\gamma > 2$ which in fact can replace (7.11). However, (7.17) contains the somewhat arbitrary spacing q, and a more natural condition for a continuous time process would restrict the size of

$$\int_1^T |r(t)| \log t \; e^{\gamma |r(t)| \log t} dt .$$

However, it is not clear how this might be done, in relation to (7.17).

CHAPTER 8

POINT PROCESSES OF UPCROSSINGS

The extremal theory of normal processes, as developed in Chapter 7, is based mainly on the asymptotic independence of maxima over several separate intervals of constant length, and on the form of the tail distribution of the maximum over one such interval. In the proofs in Chapter 7 we made use of upcrossings, and of the obvious fact that the maximum exceeds u if there is at least one upcrossing of the level u. However, as was seen already in Chapter 4, upcrossings have an interest in their own right, and as we shall see here, in this continuous time setting, they contain considerable information about the local structure of the process.

This chapter is devoted to the asymptotic Poisson character of the point process formed by the upcrossings of increasingly high levels, and indeed, this requires only little more than is needed to obtain the much weaker results of Chapter 7. In our derivation we shall make substantial use of the regularity condition $\lambda_2 < \infty$, which implies that the upcrossings do not "appear in clusters" and remain separated as the level increases. In fact, Theorems 6.7 and 6.5 imply that there are only a finite number of u-points in finite intervals. Similar resluts will be established in Chapter 12, for the case $\lambda_2 = \infty$, with regular upcrossings replaced by so called ε-upcrossings.

We shall first prove asymptotic independence of the maxima over disjoint intervals, and then use this result to prove that the point processes of upcrossings of several simultaneously increasing levels tend in distribution to a sequence of successively thinned Poisson processes.

In the case of a finite fourth spectral moment this will eventually, in Chapter 9, give the joint distribution of heights and locations of the highest local maxima.

Poisson convergence of upcrossings

Corresponding to each level u we have defined $\mu = \mu(u) = \frac{1}{2\pi}\lambda_2^{1/2} e^{-u^2/2}$ to be the mean number of u-upcrossings per time unit, and, as in Chapter 7, we consider $T = T(u)$ such that $T\mu \to \tau$ as $u \to \infty$, where $\tau > 0$ is a fixed number. Let N_T be the time-normalized point process of u-upcrossings, defined by

$$N_T^*(B) = N_u(TB) = \#\{\text{u-upcrossings by } \xi(t);\ t/T \in B\},$$

for any real Borel set B, i.e. N_T^* has a point at t if ξ has a u-upcrossing at tT. Note that we define N_T^* as a point process on the entire real line, and that the only significance of the time T is that of an appropriate scaling factor. This is a slight shift in emphasis from Chapter 7, where we considered $u_T = x/a_T + b_T$ as a height normalization for the maximum over the increasing time interval $(0,T)$.

Let N be a Poisson process on the real line with intensity τ. To prove point process convergence under suitable conditions, we need to prove different forms of asymptotic independence of maxima over disjoint intervals. For the one-level result, that N_T^* converges in distribution to N, we need only the following partial independence.

LEMMA 8.1 Let $a = a_1 < b_1 < a_2 < \ldots < a_r < b_r = b$ be fixed numbers, $E_i = (Ta_i, Tb_i]$, and $M(E_i) = \sup\{\xi(t);\ Ta_i < t \le Tb_i\}$. Then, under the conditions of Theorem 7.6,

$$P\left(\bigcap_{i=1}^{r}\{M(E_i) \le u\}\right) - \prod_{i=1}^{r} P\{M(E_i) \le u\} \to 0$$

as $u \to \infty$, $T\mu \to \tau > 0$.

PROOF The proof is similar to that of Lemmas 7.4 and 7.5. Recall the construction in Lemma 7.4, and divide the real line into intervals $\ldots, I_1, I_1^*, I_2, \ldots$ of lengths $h-\varepsilon$ and ε, alternately. We can then approximate $M(E_i)$ by the maximum on the parts of the separated intervals I_k which are contained in E_i. Write n for the number of I_k^*'s which have non-empty intersection with $\bigcup_{i=1}^{r} E_i$. We at once obtain

$$0 \le P\Big(\bigcap_{i=1}^{r}\{M(\bigcup_k I_k \cap E_i) \le u\}\Big) - P\Big(\bigcap_{i=1}^{r}\{M(E_i) \le u\}\Big)$$

$$\le n\, P\{M(I_1^*) > u\},$$

where (writing $|E|$ for the length of an interval E),

$$n \sim h^{-1} \sum_{i=1}^{r} |E_i| = \frac{T}{h} \sum_{i=1}^{r} (b_i - a_i) \le \frac{T(b-a)}{h}.$$

Since Lemma 7.2(i) implies that

$$\limsup_{u\to\infty} \mu^{-1} P\{M(I_1^*) > u\} \le \varepsilon,$$

we therefore have

$$\text{(8.1)} \quad \limsup_{u\to\infty} \Big| P\Big(\bigcap_{i=1}^{r}\{M(\bigcup_k I_k \cap E_i) \le u\}\Big) - P\Big(\bigcap_{i=1}^{r}\{M(E_i) \le u\}\Big)\Big|$$

$$\le \frac{\tau\varepsilon(b-a)}{h}.$$

Now, let $q \to 0$ as $u \to \infty$ so that $qu \to 0$. The discrete approximation of maxima in terms of $\xi(jq)$, $jq \in \bigcup_k I_k \cap E_i$ is then obtained as in Lemma 7.4(ii). In fact, since there are $n + \delta$ intervals I_k which intersect $\cup E_i$ (where $|\delta| \le r$), we have

$$\text{(8.2)} \quad 0 \le P\Big(\bigcap_{i=1}^{r}\{\xi(jq) \le u,\ jq \in \bigcup_k I_k \cap E_i\}\Big) - P\Big(\bigcap_{i=1}^{r} M(\bigcup_k I_k \cap E_i) \le u\}\Big)$$

$$\le \sum_k (P\{\xi(jq) \le u,\ jq \in I_k \cap E_i\} - P\{M(I_k \cap E_i) \le u\})$$

$$= (n+\delta)o(\mu) = o(1), \quad \text{as} \quad u \to \infty,$$

by Lemma 7.3(ii).

Furthermore,

$$\text{(8.3)} \quad P\Big(\bigcap_{i=1}^{r}\{\xi(jq) \le u,\ jq \in \bigcup_k I_k \cap E_i\}\Big) - \prod_{i=1}^{r} y_i \to 0$$

where $y_i = \prod_k P\{\xi(jq) \le u,\ jq \in I_k \cap E_i\}$, the proof this time being a rephrasing of the proof of Lemma 7.5(i).

By combining (8.1), (8.2), and (8.3) we obtain

$$\limsup_{n\to\infty} \Big|P\Big(\bigcap_{i=1}^{r}\{M(E_i) \le u\}\Big) - \prod_{i=1}^{r} y_i\Big| \le \frac{\tau\varepsilon(b-a)}{h}$$

and in particular, for $i = 1, \ldots, r$,

$$\limsup_{u\to\infty} |P\{M(E_i) \le u\} - y_i| \le \frac{\tau\varepsilon(b-a)}{h}.$$

Hence, writing $x_i = P\{M(E_i) \le u\}$, we have

$$\limsup_{u\to\infty} \left|P\left(\bigcap_{i=1}^{r}\{M(E_i) \le u\}\right) - \prod_{i=1}^{r} P\{M(E_i) \le u\}\right|$$

$$\le \frac{\tau\varepsilon(b-a)}{h} + \limsup_{u\to\infty} \left|\prod_{i=1}^{r} y_i - \prod_{i=1}^{r} x_i\right|.$$

But, with $z = \max_i |y_i - x_i|$ (so that $\limsup z \le \tau\varepsilon(b-a)/h$),

$$\prod_{i=1}^{r} y_i - \prod_{i=1}^{r} x_i \le \prod_{i=1}^{r} (x_i + z) - \prod_{i=1}^{r} x_i \le rz,$$

with a similar relation holding with y_i and x_i interchanged, and hence

$$\limsup_{u\to\infty} \left|\prod_{i=1}^{r} y_i - \prod_{i=1}^{r} x_i\right| \le \frac{r\tau\varepsilon(b-a)}{h}.$$

Since ε is arbitrary, we have proved the conclusion of the lemma. □

<u>THEOREM 8.2</u> Let $u \to \infty$ and $T \sim \tau/\mu$, where $\mu = \frac{1}{2\pi}\lambda_2^{1/2} e^{-u^2/2}$, and suppose $r(t)$ satisfies (7.1) and (7.2) (or the weaker condition (7.4)). Then the time-normalized point process N_T^* of u-upcrossings converges in distribution to a Poisson process with intensity τ.

<u>PROOF</u> By the basic convergence theorem for simple point processes, Theorem A.1 (see Appendix to Part I), it is sufficient to show that, as $u \to \infty$,

(a) $E(N_T^*((a, b])) \to E(N((a, b])) = \tau(b-a)$ for all $a < b$,

and

(b) $P\{N_T^*(B) = 0\} \to P\{N(B) = 0\} = e^{-\tau\Sigma_{i=1}^{r}(b_i - a_i)}$ for all sets B of the form $\bigcup_{i=1}^{r} (a_i, b_i]$, $a_1 < b_1 < \ldots < a_r < b_r$.

Here, part (a) is trivially satisfied, since

$$E(N_T^*((a, b])) = E(N_u(Ta, Tb]) = T\mu(b-a) \to \tau(b-a).$$

For part (b), we have for the u-upcrossings,

$$P\{N_T^*(B)=0\} = P\Big(\bigcap_{i=1}^{r}\{N_T^*((a_i, b_i]) = 0\}\Big) = P\Big(\bigcap_{i=1}^{r}\{N_u(E_i) = 0\}\Big),$$

where $E_i = (Ta_i, Tb_i]$. Now it is easy to see that we can work with maxima instead of crossings, since

$$0 \le P\Big(\bigcap_{i=1}^{r}\{N_u(E_i) = 0\}\Big) - P\Big(\bigcap_{i=1}^{r}\{M(E_i) \le u\}\Big)$$

$$= P\Big(\bigcap_{i=1}^{r}\{N_u(E_i) = 0\} \cap \bigcup_{i=1}^{r}\{M(E_i) > u\}\Big)$$

$$\le \sum_{i=1}^{r} P\{\xi(Ta_i) > u\} \to 0 \quad \text{as} \quad u \to \infty,$$

and since furthermore Corollary 7.7 and Lemma 8.1 imply that

$$\lim_{u\to\infty} P\Big(\bigcap_{i=1}^{r}\{M(E_i) \le u\}\Big) = \lim_{u\to\infty} \prod_{i=1}^{r} P\{M(E_i) \le u\} = \prod_{i=1}^{r} e^{-\tau(b_i - a_i)},$$

we have proved part (b). □

One immediate consequence of the distributional convergence of N_T, is the asymptotic Poisson distribution of the number of u-upcrossings in increasing Borel sets $T \cdot B$. Since this is an important result we formulate it as a corollary.

COROLLARY 8.3 Under the conditions of Theorem 8.2, if B is any Borel set whose boundary has Lebesgue measure zero, then

(8.4) $\quad P\{N_T^*(B) = r\} \to e^{-\tau|B|}(\tau|B|)^r/r!, \; r = 0, 1, \dots ,$

as $u \to \infty$, where $|B|$ is the Lebesgue measure of B. The joint distribution of $N_T^*(B_1), \dots, N_T^*(B_n)$ corresponding to disjoint B_j (with boundaries which have Lebesgue measure zero) converges to the product of the corresponding Poisson probabilities. □

Full independence of maxima in disjoint intervals

A topic of some interest, which we have not touched upon yet, is the relationship between the intensity τ and the height u of a level for which $T\mu \to \tau$. If $T = \tau/\mu = \tau 2\pi \lambda_2^{-1/2} e^{u^2/2}$ we have

$$u^2 = 2 \log T\left(1 - \frac{\log \tau + \log(2\pi/\lambda_2^{1/2})}{\log T}\right)$$

or

$$(8.5) \qquad u = (2 \log T)^{1/2} - \frac{\log \tau + \log(2\pi/\lambda_2^{1/2})}{(2 \log T)^{1/2}} + o((\log T)^{-1/2}).$$

However, any level which differs from u by $o((\log T)^{-1/2})$ will do equally well in Theorem 8.2, and it is often convenient to use the level obtained by deleting the last term in (8.5) entirely. (The reader should check that also for this choice the relation $T\mu \to \tau$ holds.) If we write

$$(8.6) \qquad u_\tau = (2 \log T)^{1/2} - \frac{\log \tau + \log(2\pi/\lambda_2^{1/2})}{(2 \log T)^{1/2}}$$

we have, for $\tau > \tau^* > 0$,

$$(8.7) \qquad u_{\tau^*} - u_\tau = \frac{\log \tau/\tau^*}{(2 \log T)^{1/2}} \sim \frac{\log \tau/\tau^*}{u_\tau} > 0,$$

so that levels corresponding to different intensities τ, τ^* (under the same time-normalization T) become increasingly close to each other, the difference being of the order $1/u_\tau$. Note that (8.7) holds for any u_{τ^*}, u_τ which satisfy $T\mu(u_{\tau^*}) \to \tau^*$, $T\mu(u_\tau) \to \tau$, and not only for the particular choice (8.6).

Now, let $u^{(1)} \geq u^{(2)} \geq \ldots \geq u^{(r)}$ be r levels such that the point processes of upcrossings are asymptotically Poisson with intensities $0 < \tau_1 \leq \tau_2 \leq \ldots \leq \tau_r$ under one and the same time-normalization, i.e. $T\mu(u^{(i)}) \to \tau_i$, $i = 1, \ldots, r$ as $u^{(r)} \to \infty$. We shall prove the full asymptotic independence of maxima in disjoint increasing intervals under the conditions (7.1) and (7.2) (or condition (7.4) with u replaced by $u^{(r)}$) i.e.

$$(8.8) \qquad \frac{T}{q} \sum_{\varepsilon \leq kq \leq T} |r(kq)| e^{-(u^{(r)})^2/(1+|r(kq)|)} \to 0$$

for each $\varepsilon > 0$, and some $q \to 0$ such that $u^{(r)} q \to 0$.

<u>THEOREM 8.4</u> Let $u^{(1)} \geq u^{(2)} \geq \dots \geq u^{(r)} \to \infty$, $T \to \infty$, and suppose

$$T\mu(u^{(i)}) = \frac{T}{2\pi} \lambda_2^{1/2} e^{-(u^{(i)})^2/2} \to \tau_i > 0.$$

Suppose that $r(t)$ satisfies (7.1), and either (7.2) or the weaker condition (8.8). Then, for any $0 \leq a = a_1 < b_1 \leq a_2 < \dots \leq a_s < b_s = b$, $(E_i = (Ta_i, Tb_i])$,

$$P\left(\bigcap_{i=1}^{s} \{M(E_i) \leq u_{T,i}\}\right) \to e^{-\sum_{i=1}^{s} \tau_i'(b_i - a_i)} \tag{8.9}$$

where each $u_{T,i}$ is one of $u^{(1)}, \dots, u^{(r)}$ and $T\mu(u_{T,i}) \to \tau_i'$.

<u>PROOF</u> For proof it is enough to check that

$$P\left(\bigcap_{i=1}^{s} \{M(E_i) \leq u_{T,i}\}\right) - \prod_{i=1}^{s} P\{M(E_i) \leq u_{T,i}\} \to 0,$$

and this goes step by step as the proof of Lemma 8.1, with u replaced by appropriate $u_{T,i}$. We only have to make sure that one can use the same grid in the discrete approximation for each level $u^{(i)}$, and this is easy, since (8.7) implies that $u^{(i)}q - u^{(j)}q \to 0$ under the stated conditions, so that $u^{(i)}q \to 0$ if $u^{(r)}q \to 0$, (cf. the proof of Theorem 4.9). □

Upcrossings of several adjacent levels

The Poisson convergence theorem, Theorem 8.2, implies that any of the time-normalized point processes of upcrossings of levels $u^{(1)} \geq \dots \geq u^{(r)}$ are asymptotically Poisson if $T\mu(u^{(i)}) \to \tau_i > 0$ as T, $u^{(i)} \to \infty$. We shall now investigate the *dependence* between these point processes, following similar lines to those in Chapter 4.

To describe this dependence we shall represent the upcrossings as points in the plane, rather than on the line, letting the upcrossings of the level $u^{(i)}$ define a point process on a fixed line L_i as was done in Chapter 4. However, for the normal process treated in this chapter the lines $L_1, \ldots, L_r$ can be chosen to have a very simple relation to the process itself by utilizing the process

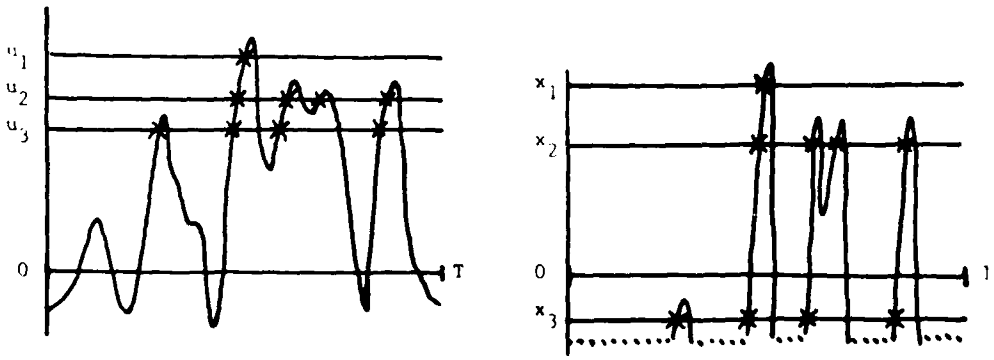
$$\xi_T(t) = a_T(\xi(tT) - b_T),$$

where time has been normalized by a factor T and height by

$$a_T = (2 \log T)^{1/2}$$
$$b_T = (2 \log T)^{1/2} + \log(\lambda_2^{1/2}/2\pi)/(2 \log T)^{1/2}$$

as usual.

Now, $\xi_T(t) = x$ if and only if $\xi(tT) = x/a_T + b_T$, and clearly the mean number of upcrossings of the level x by $\xi_T(t)$ in an interval of length h is equal to $\frac{Th}{2\pi} \lambda_2^{1/2} \exp(-(x/a_T + b_T)^2/2)$, which by (8.6) equals $h\tau(1 + o(1))$ as $T \to \infty$, with $\tau = e^{-x}$. Therefore, let $x_1 \geq x_2 \geq \cdots \geq x_r$ be a set of fixed numbers, defining horizontal lines $L_1, L_2, \ldots, L_r$, and consider the point process in the plane formed by the upcrossings of any of these lines by the process $\xi_T(t)$. Here the dependence between points on different lines is not quite as simple as it was in Chapter 4, since, unlike an exceedance, an upcrossing of a high level is not an upcrossing of a lower level and there may in fact even be more upcrossings of the higher than of the lower level; see the following diagram, which shows the relation between the upcrossings of levels $u^{(i)} = x_i/a_T + b_T$ by $\xi(t)$, and of levels x_i by $\xi_T(t)$. As is seen, local irregularities in the process $\xi(t)$ can cause the appearance of extra upcrossings of a high level, not present in the lower ones.



Let N_T^* denote the point process in the plane formed by the upcrossings of the fixed levels $x_1 \le x_2 \le \dots \le x_r$ by the process $\xi_T(t) = a_T(\xi(tT) - b_T)$, and let its components on the separate lines be $N_T^{(1)}, \dots, N_T^{(r)}$, so that

$$N_T^*(B) = \#\{\text{upcrossings in } B \text{ of } L_1, \dots, L_r \text{ by } \xi_T(t)\}$$

$$= \sum_{i=1}^{r} N_T^{(i)}(B \cap L_i),$$

for arbitrary Borel sets $B \subseteq R^2$.

We shall now prove that N_T^* converges in distribution to a point process N in the plane, which is of a type already encountered in connection with exceedances in Chapter 4. The points of N are concentrated on the lines $L_1, \dots, L_r$ and its distribution is determined by the joint distributions of its components $N^{(1)}, \dots, N^{(r)}$ on the separate lines $L_1, \dots, L_r$.

Let, as in Chapter 4, $\{\sigma_{1j};\ j = 1, 2, \dots\}$ be the points of a Poisson process $N^{(r)}$ with parameter $\tau_r = e^{-x_r}$ on L_r. Let β_j, $j = 1, 2, \dots$ be i.i.d. random variables, independent of $N^{(r)}$, with distribution defined by

$$P\{\beta_j = s\} = (\tau_{r-s+1} - \tau_{r-s})/\tau_r, \quad s = 1, \dots, r-1,$$

$$= \tau_1/\tau_r, \qquad s = r,$$

so that $P\{\beta_j \ge s\} = \tau_{r-s+1}/\tau_r$ for $s = 1, 2, \dots, r$.

Construct the processes $N^{(r-1)}, \dots, N^{(1)}$ by placing points $\sigma_{2j}, \sigma_{3j}, \dots, \sigma_{\beta_j j}$ on the $\beta_j - 1$ lines $L_{r-1}, \dots, L_{r-\beta_j+1}$, vertically above σ_{1j}, $j = 1, 2, \dots$, and finally define N to be the sum of the r processes $N^{(1)}, \dots, N^{(r)}$.

As before, each $N^{(k)}$ is Poisson on L_k, since it is obtained from the Poisson process $N^{(r)}$ by independent deletion of points, with deletion probability $1 - P\{\beta_j \ge r - k + 1\} = 1 - \tau_k/\tau_r$, and it has intensity $\tau_r(\tau_k/\tau_r) = \tau_k$. Furthermore, $N^{(k)}$ is obtained from $N^{(k+1)}$ by a binomial thinning, with deletion probability $1 - \tau_k/\tau_{k+1}$. Of course, N itself is not Poisson in the plane.

When proving the main result, that N_T^* tends in distribution to N, we need to show that asymptotically, there are not more upcrossings of a higher than of a lower level. With a convenient abuse of notation, write $N_T^{(i)}(I)$ for the number of points of $N_T^{(i)}$ with time-coordinate in I.

<u>LEMMA 8.5</u> Suppose $x_i > x_j$, and consider the point processes $N_T^{(i)}$ and $N_T^{(j)}$ of upcrossings by $\xi_T(t)$ of the levels x_i and x_j, respectively. Under the conditions of Theorem 8.2,

$$P\{N_T^{(i)}(I) > N_T^{(j)}(I)\} \to 0$$

as $T \to \infty$, for any bounded interval I.

<u>PROOF</u> By stationarity it is sufficient to prove the lemma for $I = (0,1]$. Let $I_k = (\frac{k-1}{n}, \frac{k}{n}]$, $k = 1, \ldots, n$, for fixed n and recall the notation (8.6): $u_{\tau_j} = x_j/a_T + b_T$, $\tau_j = e^{-x_j}$. Since, by Theorem 6.7 all crossings are strict, the event $\{N_T^{(i)}(I) > N_T^{(j)}(I)\}$ implies that one of the events

$$\bigcup_{k=0}^{n} \{\xi(\tfrac{kT}{n}) > u_{\tau_j}\}$$

or

$$\bigcup_{k=1}^{n} \{N_T^{(i)}(I_k) \geq 2\}$$

occurs, so that Boole's inequality and stationarity give

$$P\{N_T^{(i)}(I) > N_T^{(j)}(I)\} \leq \sum_{k=0}^{n} P\{\xi(\tfrac{kT}{n}) > u_{\tau_j}\} + \sum_{k=1}^{n} P\{N_T^{(i)}(I_k) \geq 2\}$$

$$= (n+1)P\{\xi(0) > u_{\tau_j}\} + n\,P\{N_T^{(i)}(I_1) \geq 2\}.$$

Obviously, $(n+1)P\{\xi(0) > u_{\tau_j}\} \to 0$, while by Corollary 8.3,

$$P\{N_T^{(i)}(I_1) \geq 2\} \to 1 - e^{-\tau_i/n} - \frac{\tau_i}{n} e^{-\tau_i/n},$$

which implies

$$\limsup_{T\to\infty} P\{N_T^{(i)}(I) > N_T^{(j)}(I)\} \leq n(1 - e^{-\tau_i/n} - \frac{\tau_i}{n} e^{-\tau_i/n}).$$

Since n is arbitrary and

$$n(1 - e^{-\tau/n} - \frac{\tau}{n} e^{-\tau/n}) \to 0$$

as $n \to \infty$ $(\tau > 0)$, this proves the lemma. □

THEOREM 8.6 Suppose that $r(t)$ satisfies (7.1) and (7.2) (or, more generally (8.8)), let $\tau_1 < \tau_2 < \ldots < \tau_k$ be real positive numbers, and let N_T^* be the point process of upcrossings of the levels $x_1 > x_2 > \ldots > x_r$ $(\tau_i = e^{-x_i})$ by the normalized process $\xi_T(t) = a_T(\xi(tT) - b_T)$ represented on the lines $L_1, \ldots, L_r$. Then, as $T \to \infty$, N_T^* tends in distribution to the point process N in the plane, described above, with points on the horizontal lines L_i, $i = 1, \ldots, r$, generated by a Poisson process $N^{(r)}$ on L_r with intensity τ_r, and a sequence of successive binomial thinnings $N^{(k)}$ with deletion probabilities $1 - \tau_k/\tau_{k+1}$, $k = 1, \ldots, r-1$.

PROOF This follows similar lines as the proof of Theorem 4.11, in that one has to show that

(a) $E(N_T^*(B)) \to E(N(B))$ for all sets B of the form $(a, b] \times (\alpha, \beta]$, $0 < a < b$, $\alpha < \beta$, and

(b) $P\{N_T^*(B) = 0\} \to P\{N(B) = 0\}$ for all sets B which are finite unions of disjoint sets of this form.

Here, if $B = (a, b] \times (\alpha, \beta]$ and $(\alpha, \beta]$ contains exactly the lines $L_s, \ldots, L_t$,

$$E(N_T^*(B)) = E\left(\sum_{k=s}^{t} N_T^{(k)}((a, b])\right) = \sum_{k=s}^{t} T(b-a)\mu(u^{(k)}) \to (b-a)\sum_{k=s}^{t} \tau_k$$
$$= E(N(B))$$

so that (a) is satisfied.

To prove (b), as in the proof of Theorem 4.11 write B in the form

$$B = \bigcup_{k=1}^{m} C_k = \bigcup_{k=1}^{m}\left((a_k, b_k] \times \bigcup_{j=1}^{j_k} (\alpha_{kj}, \beta_{kj}]\right),$$

where $(a_k, b_k]$ and $(a_\ell, b_\ell]$ are disjoint for $k \neq \ell$. For each k, let m_k be the index of the lowest L_j that intersects C_k, i.e. $L_{m_k} \cap C_k \neq \emptyset$, $L_j \cap C_k = \emptyset$ for $j > m_k$. Then clearly, if $N_T^{(m_k)}((a_k, b_k]) = 0$ then either $N_T(C_k) = 0$ or there is an index $i < m_k$ such that $N_T^{(i)}((a_k, b_k]) > 0$, i.e. in $(a_k, b_k]$ there are more upcrossings of a higher than of a lower level. Since obviously $N_T(C_k) = 0$ implies $N_T^{(m_k)}((a_k, b_k]) = 0$,

$$0 \leq P\left(\bigcap_{k=1}^{m} \{N_T^{(m_k)}((a_k, b_k]) = 0\}\right) - P\{N_T(B) = 0\} \to 0,$$

since the difference is bounded by the probability that some higher level has more upcrossings than a lower one, which tends to zero by Lemma 8.5.

But

$$\{N_T^{(m_k)}((a_k, b_k]) = 0\} \cap \{\xi_T(a_k) \leq x_{m_k}\} = \{M((Ta_k, Tb_k]) \leq u_{T,k}\}$$

where $u_{T,k} = x_{m_k}/a_T + b_T$, so that Theorem 8.4 implies that

$$\lim_{T\to\infty} P\left(\bigcap_{k=1}^{m} \{N^{(m_k)}((a_k, b_k]) = 0\}\right) = \lim_{T\to\infty} P\left(\bigcap_{k=1}^{m} \{M((Ta_k, Tb_k]) \leq u_{T,k}\right)$$
$$= \prod_{k=1}^{m} e^{-\tau_k'(b_k - a_k)},$$

where $\tau_k' = \tau_{m_k} = e^{-x_{m_k}}$. Clearly this is just $P\{N(B) = 0\}$, and thus the proof of part (b) is complete. □

<u>COROLLARY 8.7</u> Let $\{\xi(t)\}$ satisfy the conditions of Theorem 8.6, and let $B_1, \ldots, B_r$ be real Borel sets, whose boundaries have Lebesgue measure zero. Then, for integers $m_j^{(k)}$,

$$P\{N_T^{(k)}(B_j) = m_j^{(k)},\ j = 1, \ldots, s,\ k = 1, \ldots, r\}$$
$$\to P\{N^{(k)}(B_j) = m_j^{(k)},\ j = 1, \ldots, s,\ k = 1, \ldots, r\}.$$

For example, for disjoint B_1 and B_2,

$$P\{N_T^{(1)}(B_1) = m_1^{(1)},\ N_T^{(1)}(B_2) = m_2^{(1)},\ N_T^{(2)}(B_2) = m_2^{(2)}\}$$

$$\to e^{-\tau_1|B_1|}\ \frac{(\tau_1|B_1|)^{m_1^{(1)}}}{m_1^{(1)}!} \cdot e^{-\tau_2|B_2|}\ \frac{(\tau_2|B_2|)^{m_2^{(2)}}}{m_2^{(2)}!}$$

$$\cdot \binom{m_2^{(2)}}{m_2^{(1)}} \left(\tau_1/\tau_2\right)^{m_2^{(1)}} \left(1-\tau_1/\tau_2\right)^{m_2^{(2)}-m_2^{(1)}}.$$

CHAPTER 9

LOCATION OF MAXIMA

So far, we have examined the extremal properties of a continuous process $\xi(t)$ by sections at certain (increasing) levels. Even if this gives perfect information about the height of the global maximum of the process, it does not directly tell us where this maximum occurs or how it is related to possible lower *local maxima*.

The maximum of $\xi(t)$, $0 \leq t \leq T$, is certainly attained at some point in $[0,T]$. However, the maximum level may be reached many times, or even infinitely often. But there will – by continuity of $\xi(t)$ – be a first occasion on which $\xi(t)$ attains its maximum in $[0,T]$, and we denote this by $L(T)$.

We state the first result concerning $L(T)$ as a lemma, though it is rather obvious.

LEMMA 9.1 $L(T)$ is a r.v. For $0 \leq t \leq T$, $P\{L(T) \leq t\} =$ $= P\{M(0,t) \geq M(t,T)\}$.

PROOF Both statements follow from the equivalence of the events $\{L(T) \leq t\}$ and $\{M(0,t) \geq M(t,T)\}$, the latter being measurable since $M(0,t)$ and $M(t,T)$ are r.v.'s. □

The distribution of $L(T)$ can have a jump at 0 and at T as simple examples (such as the process $\xi(t) = A\cos(t-\phi)$ with $T < 2\pi$) show. However, a simple condition precludes the possibility of any other jumps in the distribution of $L(T)$, and this is generally satisfied except in "degenerate" or "deterministic" cases.

Specifically we will say that $\xi(t)$ has a *derivative in probability* at t_0 if there exists a r.v. η such that

$$\frac{\xi(t_0+h) - \xi(t_0)}{h} \to \eta \text{ in probability as } h \to 0.$$

Clearly if ξ has a q.m. or probability-one derivative, it has a derivative in probability (with the same value).

<u>THEOREM 9.2</u> Suppose that $\xi(t)$ has a derivative in probability at t (where $0 < t < T$), and that the distribution of this derivative is continuous at zero. Then $\Pr\{L(T) = t\} = 0$.

<u>PROOF</u> Let η denote the derivative in probability at t. Clearly

$$\{L(T) = t\} \subset \{\frac{\xi(t) - \xi(t-h)}{-h} \leq 0\} \cap \{\frac{\xi(t) - \xi(t+h)}{h} \geq 0\}$$

for all $h > 0$ such that $0 \leq t - h$ and $t + h \leq T$.

Now $(\xi(t+h) - \xi(t))/h \to \eta$ in probability as $h \downarrow 0$ and there exists a sequence $\{h_n\}$ such that $(\xi(t+h_n) - \xi(t))/h_h \to \eta$ with probability one. By considering a subsequence of $\{h_n\}$ we may also arrange that $(\xi(t-h_n) - \xi(t))/(-h_n) \to \eta$ with probability one, i.e. on a set B with $P(B) = 1$. We see that $\eta = 0$ on $\{L(T) = t\} \cap B$, i.e. $\{L(T) = t\} \cap B \subset \{\eta = 0\}$ and hence

$$P\{L(T) = t\} \leq P(\{L(T) = t\} \cap B) + P(B^c) \leq P\{\eta = 0\} + P(B^c) = 0,$$

when η has a continuous distribution. □

Turning to stationary processes, one may be tempted to conjecture that if $\xi(t)$ is stationary, then $L(T)$ is uniformly distributed over $(0,T)$. For example this is so if $\xi(t) = A\cos(t-\phi)$, with ϕ uniformly distributed over $(0,2\pi]$, for $T = 2\pi$. (If A is Rayleigh distributed and independent of ϕ, $\xi(t)$ of course is normal.)

If $T < 2\pi$, there is a positive probability of L being 0 or T, and $L(T)$ is not strictly uniform. However, its distribution is still uniform *between* 0 and T as a simple calculation shows.

In general, however, L need *not* be uniform in the open interval $(0,T)$, not even if $\xi(t)$ is normal and stationary. As an example of this, let ϕ_1, ϕ_2, A_1, and A_2 be independent, with ϕ_1 and ϕ_2 uniform over $(0,2\pi]$, and with A_1 and A_2 Rayleigh distributed, and put $\xi(t) = A_1\cos(t-\phi_1) + A_2\cos(100t-\phi_2)$. Then $\xi(t)$ is a stationary normal process, and (e.g. by drawing a picture) it can be seen that if $A_1 < A_2$, $\phi_1 \in (3\pi/2, 2\pi]$, and $\phi_2 \in (\pi/4, 3\pi/4)$ then the maximum of

$\xi(t)$ over $[0,\pi/2]$ is attained in the interval $(0,\pi/100]$. Hence $P\{L(\pi/2) \leq \pi/100\} \geq P\{A_1 < A_2,\ 3\pi/2 < \phi_1 \leq 2\pi,\ \pi/4 < \phi_2 \leq 3\pi/4\} =$ $= (1/2)(1/4)(1/4) = 1/32 > 1/50 = (\pi/100)/(\pi/2)$, and $L(\pi/2)$ can not be uniform over $(0,\pi/2)$.

However, for a stationary *normal* process the distribution of L is always *symmetric* in the entire interval $[0,T]$, and possible jumps at 0 and T are equal in magnitude. This follows from the reversibility of a stationary normal process in the sense that $\{\xi(-t)\}$ has the same distribution as $\{\xi(t)\}$.

One method of removing boundaries, like 0 and T, is to let them disappear to infinity, and one may ask whether $L = L(T)$ might be asymptotically uniform as $T \to \infty$. For normal processes, this is a simple consequence of the asymptotic independence of maxima over disjoint intervals, as was previously mentioned. We state these results here, as simple consequences of Theorem 8.4.

THEOREM 9.3 Let $\{\xi(t)\}$ be a stationary normal process (standardized as usual) with $\lambda_2 < \infty$, and suppose that $r(t)\log t \to 0$ as $t \to \infty$. Then,

$$P\{L(T) \leq \ell T\} \to \ell \quad \text{as} \quad T \to \infty \quad (0 \leq \ell \leq 1).$$

PROOF With the usual notation, if $0 \leq \ell \leq 1$, $\ell^* = 1 - \ell$, and

$$X_T = a_{\ell T}(M(0,\ell T) - b_{\ell T}),$$

$$Y_T = a_{\ell^* T}(M(\ell T,T) - b_{\ell^* T}),$$

where a's and b's are given by (7.8), then

$$P\{X_T \leq x,\ Y_T \leq y\} \to \exp\{-e^{-x} - e^{-y}\}$$

$(x \geq 0,\ y \geq 0)$ as $T \to \infty$. Furthermore

$$P\{L(T) \leq \ell T\} = P\{M(0,\ell T) \geq M(\ell T,T)\}$$

$$= P\{X_T - \frac{a_{\ell^* T}}{a_{\ell T}} Y_T \geq a_{\ell T}(b_{\ell^* T} - b_{\ell T})\}$$

where $a_{\lambda^* T}/a_{\lambda T} \to 1$ and $a_{\lambda T}(b_{\lambda^* T} - b_{\lambda T}) \to \log(\lambda^*/\lambda)$. As $T \to \infty$ the above probability tends to

$$P\{X - Y \geq \log \lambda^*/\lambda\}$$

where X and Y are independent r.v.'s with common d.f. $\exp(-e^{-x})$, and an evaluation of this probability yields the desired value [illegible]. □

Height and location of local maxima

One consequence of Theorem 9.3 is that asymptotically the global maximum is attained in the interior $(0,T)$ and thus also is a local maximum. For sufficiently regular processes one might consider also smaller, but still high, *local maxima*, which are separated from $L(T)$.

We first turn our attention to continuously differentiable normal processes which are twice differentiable in quadratic mean.

In analogy to the development in Chapter 4, we shall consider the point process in the plane, which is formed by the suitably transformed local maxima of $\xi(t)$. (Note that since the process $\xi(t)$ is continuous, the path of $a_T(\xi(t) - b_T)$ is also continuous, and although its visits to any bounded rectangle $B \subset R^2$ are approximately Poisson in number, they are certainly not *points*.)

Suppose $\xi(t)$, $0 \leq t \leq T$ has local maxima at the points s_i with height $\xi(s_i)$. Let a_T and b_T be the normalizing constants defined by (7.8), and define a point process in the plane by putting points at $(T^{-1}s_i,\ a_T(\xi(s_i) - b_T))$. We recall from Chapter 8 that asymptotically the upcrossings of the fixed level x by $a_T(\xi(t) - b_T)$ form a Poisson process with intensity $\tau = e^{-x}$ when time is normalized to t/T, and that an upcrossing of a level x is accompanied by an upcrossing of the higher level y with a probability $e^{-y}/e^{-x} = e^{-(y-x)}$.

When investigating the Poisson character of local maxima, a question of some interest is to what extent high level upcrossings and high local maxima can replace each other. Obviously there must be at least one local maximum between an upcrossing of a certain level u and the next downcrossing of the same level, so that, loosely speaking, there are at least as many high maxima as there are high upcrossings. As will now be seen there are, with high probability, no more. In fact we shall see that this is true even when $T \to \infty$ in such a way that $T\mu =$ $= T \frac{1}{2\pi} \lambda_2^{1/2} e^{-u^2/2}$ converges.

First recall the notation from Chapter 6,

$$N_u'(a,b] = \#\{s_i \in (a,b];\ \xi(s_i) > u\},$$

$$N_u'(T) = N_u'(0,T].$$

<u>LEMMA 9.4</u> If $\lambda_4 < \infty$ and $T \sim \tau/\mu = \tau\, 2\pi\, \lambda_2^{-1/2} e^{u^2/2}$, then

(i) $\quad E(N_u'(T)) \to \tau$

and

(ii) $\quad P\{|N_u'(T) - N_u(T)| \geq 1\} \to 0$

as $u \to \infty$.

<u>PROOF</u> First note that at least one of the following events occur,

$$\{N_u'(T) \geq N_u(T)\} \quad \text{or} \quad \{\xi(T) > u\}$$

and that in the latter case, $N_u'(T) \geq N_u(T) - 1$. Therefore

$$P\{|N_u'(T) - N_u(T)| \geq 1\} \leq E(|N_u'(T) - N_u(T)|)$$

$$\leq E(N_u'(T) - N_u(T)) + 2P\{\xi(T) > u\},$$

and since $E(N_u(T)) = \tau$ and $P\{\xi(T) > u\} \to 0$, (ii) is a direct consequence of (i). But $E(N_u'(T))$ is given by (6.18) where we can use $1 - \Phi(x) \sim \phi(x)/x$ as $x \to \infty$, so that for some constant K,

$$\frac{T}{2\pi}\left(\frac{\lambda_4}{\lambda_2}\right)^{1/2}\left(1-\Phi\left(u\left(\frac{\lambda_4}{\lambda_4-\lambda_2^2}\right)^{1/2}\right)\right) \le K\,\frac{T}{u}\,\phi\left(u\left(\frac{\lambda_4}{\lambda_4-\lambda_2^2}\right)^{1/2}\right) \le K\,\frac{T}{u}\,\phi(u) \to 0$$

while

$$\frac{T}{2\pi}\left(\frac{\lambda_4}{\lambda_2}\right)^{1/2}\lambda_2\left(\frac{2\pi}{\lambda_4}\right)^{1/2}\phi(u)\,\Phi\left(u\left(\frac{\lambda_2}{(\lambda_4-\lambda_2^2)^{1/2}}\right)\right)$$

$$= \frac{T}{2\pi}\lambda_2^{1/2}e^{-u^2/2}(1+o(1)) \to \tau$$

as $u \to \infty$, which proves (i). □

<u>THEOREM 9.5</u> Suppose the standardized stationary normal process $\{\xi(t)\}$ has continuously differentiable sample paths and is quadratic mean differentiable (i.e. $\lambda_4 < \infty$), and suppose that $r(t)\log t \to 0$ as $t \to \infty$. Then the point process N_T' of normalized local maxima $(s_i/T, a_T(\xi(s_i)-b_T))$ converges in distribution to a Poisson process N' in the plane with intensity measure equal to the product of Lebesgue measure and that defined by the increasing function $-e^{-x}$.

<u>PROOF</u> By Theorem A.1 it is enough to show that

(a) $E(N_T'(B)) \to E(N'(B)) = (b-a)(e^{-\alpha}-e^{-\beta})$ for any set B of the form $(a,b]\times(\alpha,\beta]$, $0 < a < b$, $\alpha < \beta$, and

(b) $P\{N_T'(B)=0\} \to P\{N'(B)=0\}$ for sets B which are finite unions of sets of this form.

To prove (a), we use Lemma 9.4 (i). Then, with $u^{(1)} = \frac{\beta}{a_T}+b_T$, $u^{(2)} = \frac{\alpha}{a_T}+b_T$,

$$E(N_T'(B)) = E(N'_{u^{(2)}}(Ta,Tb)) - E(N'_{u^{(1)}}(Ta,Tb)) \to$$

$$\to (b-a)e^{-\alpha} - (b-a)e^{-\beta} = (b-a)(e^{-\alpha}-e^{-\beta}),$$

since $T\mu(u^{(i)}) \to e^{-\beta}, e^{-\alpha}$ for $i = 1, 2$.

Part (b) is a consequence of Lemma 9.4 (ii) and the multilevel upcrossing theorem, Theorem 8.6. Let $N_u(I)$, as before, denote the number of u-upcrossings by $\xi(t)$, $t \in I$, and write the set B in the form $\bigcup_j E_j \times F_j$, where $E_j = (a_j, b_j]$ are disjoint and each F_j is a finite union of disjoint intervals. Suppose first that there is only one set E_j, i.e. $B = E \times \bigcup_k G_k$, where $G_k = (\alpha_k, \beta_k]$, and write $u^{(2k-1)} = \beta_k/a_T + b_T$, $u^{(2k)} = \alpha_k/a_T + b_T$. According to Lemma 9.4 (ii) asymptotically every upcrossing of the high level u is accompanied by one (and no more) local maximum above that level, and hence

$$P\{N_T'(E \times G_k) = 0\} = P\{N'_{u^{(2k)}}(Ta,Tb) = N'_{u^{(2k-1)}}(Ta,Tb)\}$$

$$= P\{N_{u^{(2k)}}(Ta,Tb) = N_{u^{(2k-1)}}(Ta,Yb)\} + o(1).$$

By Theorem 8.6, with $\tau_{2k} = e^{-\alpha_k}$, $\tau_{2k-1} = e^{-\beta_k}$,

$$P\{N_{u^{(2k)}}(Ta,Tb) = N_{u^{(2k-1)}}(Ta,Tb)\}$$

$$\to \sum_{j=0}^{\infty} e^{-\tau_{2k}(b-a)} \frac{(\tau_{2k}(b-a))^j}{j!} \left(\frac{\tau_{2k-1}}{\tau_{2k}}\right)^j$$

$$= e^{-(\tau_{2k} - \tau_{2k-1})(b-a)} = e^{-(b-a)(-e^{-\beta_k} - (-e^{-\alpha_k}))}$$

$$= P\{N'(E \times G_k) = 0\}.$$

By slightly extending the argument we obtain

$$P\{N_T'(E \times \bigcup_k G_k) = 0\} = P\left(\bigcup_k \{N_{u^{(2k)}}(Ta,Tb) = N_{u^{(2k-1)}}(Ta,Tb)\}\right) + o(1)$$

$$\to e^{-(b-a)\sum_k (e^{-\alpha_k} - e^{-\beta_k})}$$

$$= P\{N'(E \times \bigcup_k G_k) = 0\},$$

and we have proved part (b) for sets B of the simple form $B = E \times \bigcup_k G_k$. The general proof of part (b) is only notationally more complex. $\square$

Location of the largest maxima

The limiting Poisson process in Theorem 9.5 has exactly the same distributions as that in Theorem 4.17 for $\xi(t)$ normal, since $\log G(s) = -e^{-s}$ in this case. This means that all consequences that can be drawn from that theorem about asymptotic properties of the normalized point process $a_n(\xi_i - b_n)$ also carries over to the normalized point process of local maxima $a_T(\xi(s_i) - b_T)$.

As an example we shall use Theorem 9.5 to give the simultaneous distribution of location and height of the two largest local maxima of $\xi(t)$, $t \in (0,T]$. Let $M_1(T)$ be the highest and $M_2(T)$ the second highest local maximum, and $L_1(T)$, $L_2(T)$ their location.

THEOREM 9.6 Suppose $\{\xi(t)\}$ satisfies the hypotheses of Theorem 9.4. Then

(9.1) $$P\{a_T(M_1(T) - b_T) \le x_1,\ L_1(T) \le \ell_1 T,\ a_T(M_2(T) - b_T) \le x_2,\ L_2(T) \le \ell_2 T\} \to \ell_1 \ell_2\, e^{-e^{-x_2}}(1 + e^{-x_2} - e^{-x_1})$$

as $T \to \infty$, for $0 \le \ell_1, \ell_2 \le 1$, $x_2 \le x_1$.

PROOF The asymptotic distribution of the *heights* of the two highest local maxima,

$$P\{a_T(M_1(T) - b_T) \le x_1,\ a_T(M_2(T) - b_T) \le x_2\} \to e^{-e^{-x_2}}(1 + e^{-x_2} - e^{-x_1}),$$

is a direct consequence of Theorem 4.14, formula (4.20), and the observation above that the limiting point process of normalized local maxima $(s_i/T,\ a_T(\xi(s_i) - b_T))$, $0 \le s_i \le T$, is the same as that of a normalized sequence of independent normal r.v.'s $(i/n,\ a_n(\xi_i - b_n))$, $i = 1, \ldots, n$.

But also the location of the local maxima can be obtained in this way. Suppose e.g. $\ell_1 < \ell_2$, and write I, J, K for the intervals $(0,\ell_1 T)$, $(\ell_1 T, \ell_2 T)$, $(\ell_2 T, T)$, respectively. With $u^{(1)} = x_1/a_T + b_T$,

$u^{(2)} = x_2/a_T + b_T$ the event in (9.1) can be expressed in terms of the highest and second highest local maxima over I, J, K as

$$\{M_1(I) \le u^{(1)},\ M_2(I) \le u^{(2)},\ M_1(J) \le u^{(2)},\ M_1(J) \le M_1(J),$$
$$M_1(K) \le M_2(I \cup J)\}$$

and the limit of the probability of this event, when expressed in terms of the point process N_T' of local maxima, is again the same as it would be for the point process of normalized independent r.v.'s. For such a process obviously $L_1(T)/T$ and $L_2(T)/T$ are independent and uniformly distributed over $(0,1)$ and independent of the heights of the maxima, which proves the theorem. □

Maxima under more general conditions

We have investigated the local maxima under the rather restrictive assumption that $\xi(t)$ is twice differentiable (in quadratic mean), i.e. $\lambda_4 < \infty$. If $\lambda_4 = \infty$ the mean number of zeros of $\xi'(t)$ is infinite, by Rice's formula, and in fact infinity close to every local maximum there are infinitely many more, which precludes the possibility of a Poisson type limit theorem for the locations of local maxima.

One way of getting around this difficulty is to exclude from further considerations a small interval around each high maximum, starting with the highest. To be more precise, let

$$M_1(T) = \sup\{\xi(t);\ t \in (0,T)\}$$

be the global maximum, and

$$L_1(T) = \inf\{t > 0;\ \xi(t) = M_1(T)\}$$

its location. For $\varepsilon > 0$ an arbitrary but fixed constant, let $I_1 = (0, L_1(T) - \varepsilon) \cup (L_1(T) + \varepsilon, T)$, and define

$$M_{2,\varepsilon}(T) = \sup\{\xi(t);\ t \in I_1\}$$

$$L_{2,\varepsilon}(T) = \inf\{t \in I_1;\ \xi(t) = M_{2,\varepsilon}(T)\}.$$

Proceeding recursively, with

$$I_k = I_{k-1} \cap [L_{k-1,\varepsilon}(T) - \varepsilon,\ L_{k-1,\varepsilon}(T) + \varepsilon]^c,$$

we get a sequence $M_{k,\varepsilon}(T)$, $L_{k,\varepsilon}(T)$, $M_{1,\varepsilon}(T) = M_1(T)$, $L_{1,\varepsilon}(T) = L_1(T)$ of heights and locations of *ε-maxima*, and there are certainly only a finite number of those in any finite interval. In fact, it is not difficult to relate these variables to the point processes of upcrossings (in the same way as regular local maxima can be replaced by upcrossings of high levels if $\lambda_4 < \infty$) and thereby obtain the following Poisson limit theorem, the proof of which is omitted.

<u>THEOREM 9.7</u> Suppose $\{\xi(t)\}$ is a standardized normal process with $\lambda_2 < \infty$ and with $r(t)\log t \to 0$ as $t \to \infty$. Then the point process $N_T^{(\varepsilon)}$ of normalized ε-maxima $(L_{i,\varepsilon}(T)/T,\ a_T(M_{i,\varepsilon}(T) - b_T))$ converges in distribution to the same Poisson process N' in the plane as in Theorem 9.5. □

Note that the limiting properties are independent of the ε chosen. We shall return to processes with even more irregularity in Chapter 12.

CHAPTER 10

SAMPLE PATH PROPERTIES AT UPCROSSINGS

Our main concern in previous chapters has been the distribution of the number and location of upcrossings of one or several adjacent levels and of high local maxima. For instance we know from Theorem 8.6 and relation (8.7) that for a standard normal process each upcrossing of the high level $u = u_\tau$ with a probability $p = \tau^*/\tau$ is accompanied by an upcrossing also of the level

$$u_{\tau^*} = u - \frac{\log p}{u},$$

asymptotically independently of all other upcrossings of u_τ and u_{τ^*}.

In order to throw further light on the conclusions and proofs in previous chapters we will now introduce some new concepts which give more precise information about the structure of the sample paths of $\{\xi(t)\}$ near upcrossings of a level u. Perhaps a word of warning is appropriate here, that we will need some slightly more difficult arguments that have been encountered so far.

We assume, as we did in Chapters 7 - 9, that $\{\xi(t)\}$ is a stationary normal process with $E(\xi(t)) = 0$, $E(\xi^2(t)) = 1$ and covariance function $r(\tau)$ satisfying

(10.1) $r(\tau) = 1 - \lambda_2\tau^2/2 + o(\tau^2)$ as $\tau \to 0$.

With a slightly more restrictive assumption,

(10.2) $-r''(\tau) = \lambda_2 + o(|\log|\tau||^{-a})$ as $\tau \to 0$

for some $a > 1$, we can assume that $\{\xi(t)\}$ has continuously differentiable sample paths - see Cramér and Leadbetter (1967), Section 9.5 - and we will do so since it serves our purposes of illustration. We also assume throughout this chapter that for each choice of distinct non-zero points $s_1, \ldots, s_n$, the distribution of $\xi(0)$, $\xi'(0)$, $\xi(s_1), \ldots, \xi(s_n)$ is non-singular. (A sufficient condition for this is that the spectral distribution $F(\lambda)$ has a continuous component; see Cramér and Leadbetter (1967), Section 10.6.

Since $\lambda_2 = \int \lambda^2 dF(\lambda) < \infty$ the number of upcrossings of the level u in any bounded interval has a finite expectation and so will be finite with probability one. Let

$$\ldots < t_{-1} < t_0 < t_1 < t_2 < \ldots$$

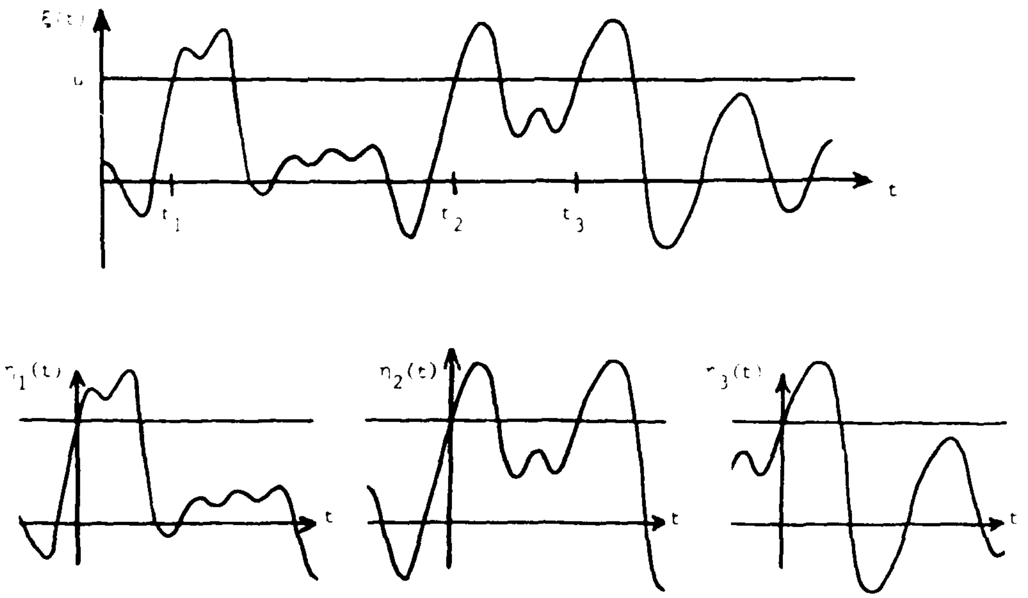
with $t_0 < 0 < t_1$ be the locations of the upcrossings of u by $\{\xi(t)\}$, and note that $t_k \to \pm\infty$ as $k \to \pm\infty$. As before, we denote by N_u the point process of upcrossings with events at $\{t_k\}$.

In order to retain information about $\{\xi(t)\}$ near its upcrossings, we now attach to each t_k a mark η_k which is a function defined as

$$\eta_k(t) = \xi(t_k + t).$$

Thus, the mark $\{\eta_k(t)\}$ is the entire sample function of $\{\xi(t)\}$ translated back by the distance t_k.

In particular $\eta_k(0) = u$, while for small t-values $\eta_k(t)$ describes



the behaviour of the ξ-process in the immediate vicinity of its k-th u-upcrossing t_k. Of course, any of the marks, say $\{\eta_0(t)\}$, contains perfect information about all the upcrossings and all other marks, so different $\{\eta_k(t)\}$ are totally dependent.

The marks are furthermore constrained by the requirement that t_1 is the first upcrossing of u after zero, t_2 the second and so on, which suggests that the different marks $\ldots, \{\eta_1(t)\}, \{\eta_2(t)\}, \ldots$ are not identically distributed.

One would feel inclined to look for the distribution of the mark at an arbitrary upcrossing without respect to its location. Intuitively, the distribution of $\eta_k(t)$ is the conditional distribution of the process at time $t_k + t$ given that there is an upcrossing of the level u at some time t_k. Two difficulties are involved here. One is, as was previously noted, that the marks are not identically distributed, which is somewhat incidental and due to the choice of time origin. The other difficulty is that the event that there is an upcrossing of u at t has probability zero if t is a fixed point. It is therefore not at all clear how one should define the conditional probabilities.

In point process theory, the abovementioned difficulties are resolved by the use of *Palm distributions* (or *Palm measures*), which formalize the notion of conditional distributions given that the point process has a point at a specified time τ. We shall use a similar approach here to obtain a Palm distribution of the mark at an arbitrary upcrossing, and then examine this distribution in some detail.

Palm distributions of the marks at upcrossings

The point process N_u on the real line formed by the upcrossings of the level u by $\{\xi(t)\}$ is stationary and without multiple events, i.e. the joint distribution of $N_u(t + I_j)$, $j = 1, \ldots, n$ does not depend on t, and $N_u(\{t\})$ is either 0 or 1; (here $t + I_j$ is the set I_j

translated by an amount t, i.e. $t + I_j = \{t + s;\ s \in I_j\}$. Further, N_u is jointly stationary with $\{\xi(t)\}$, i.e. the distribution of $N_u(t + I_j)$, $\xi(t + s_j)$, $j = 1, \ldots, n$ does not depend on t.

We shall now define the Palm distribution of the marks in accordance with the intuitive meaning of "the behaviour of $\xi(t_k + t)$ regardless of the value of t_k". Let s_j, $j = 1, \ldots, n$ and y_j, $j = 1, \ldots, n$ be fixed numbers. For each upcrossing, t_k, it is then possible to check whether $\xi(t_k + s_j) \le y_j$, $j = 1, \ldots, n$ or not. Now, starting from the point process N_u, form a new process $\tilde{N}_u$ by deleting all points t_k in N_u that do not satisfy $\xi(t_k + s_j) \le y_j$, $j = 1, \ldots, n$. The relative number of points in this new point process tells us how likely it is (for the particular sample function) that a point t_k in N_u is accompanied by a mark $\eta_k(t)$, satisfying $\eta_k(s_j) \le y_j$, $j = 1, \ldots, n$.

<u>DEFINITION 10.1</u> The finite-dimensional Palm distributions P_0^u of the mark $\{\eta_0(t)\}$ are given by

$$P_0^u(\eta_0(s_j) \le y_j,\ j = 1, \ldots, n) = \frac{E(\tilde{N}_u(1))}{E(N_u(1))}$$

$$= \frac{E(\#\{t_k \in (0,1);\ \xi(t_k + s_j) \le y_j,\ j = 1, \ldots, n\})}{E(\#\{t_k \in (0,1)\})} \tag{10.3}$$

The Palm distributions P_0^u of the mark $\{\eta_\nu(t)\}$ for any integer ν, are given by

$$P_0^u(\eta_\nu(s_j) \le y_j,\ j = 1, \ldots, n)$$

$$= \frac{E(\#t_k \in (0,1);\ \xi(t_{k+\nu} + s_j) \le y_j,\ j = 1, \ldots, n\})}{E(\#\{t_k \in (0,1)\})},$$

and the joint Palm distribution of $\{\eta_0(t)\}$ and $\{\eta_\nu(t)\}$ is defined similarly. □

Relation (10.3) defines a consistent family of finite-dimensional distributions for a certain stochastic process, which will be studied in more detail later in this chapter. But first we give some more connections between Palm distributions and the marks at upcrossings.

Palm probabilities can also be obtained as limits of ordinary conditional probabilities given a point, i.e. an upcrossing, not exactly at 0, but somewhere nearby. Let t_0 be the last u-upcrossing for $\xi(t)$ prior to 0. Then

$$P_0^u\{\eta_0(s_j) \le y_j,\ j=1,\ \dots,\ n\}$$

$$= \lim_{h \downarrow 0} P\{\xi(t_0+s_j) \le y_j,\ j=1,\ \dots,\ n \mid -h < t_0 < 0\}, \tag{10.4}$$

where it may be shown that the limit (10.4) exists, and equals the ratio (10.3). In fact, (10.4) can be taken as a definition of the Palm distributions, an approach which was taken by Kac and Slepian (1959), who also termed it the *horizontal window* conditioning of crossings, indicating that the sample path $\xi(t)$ has to pass through a horizontal window $\{(t,\ u);\ -h \le t \le 0\}$. This is in contrast to *vertical window* conditioning which requires that $u \le \xi(0) \le u+h$, $\xi'(0) > 0$, so that the process has to pass through a vertical window $\{(0,\ x);\ u \le x \le u+h\}$ with positive slope.

The following important proposition relates the Palm distribution to the empirical distribution of the values of $\xi(t_k+t)$ when t_k runs through the set of all upcrossings of the level u. It states that the Palm distributions in an ergodic process can actually be observed by considering the marks over an increasing interval.

<u>PROPOSITION 10.2</u> If the process $\{\xi(t)\}$ is ergodic, then with probability one

$$P_0^u\{\eta_\nu(s_j) \le y_j,\ j=1,\ \dots,\ n\}$$

$$= \lim_{T\to\infty} \frac{\#\{t_k \in (0,T);\ \xi(t_{k+\nu}+s_j) \le y_j,\ j=1,\ \dots,\ n\}}{\#\{t_k \in (0,T)\}}. \quad \square \tag{10.5}$$

The convergence of the empirical finite-dimensional distributions in (10.5) to the Palm distributions can be extended to empirical distributions of functionals such as the excursion time, i.e. the time from the upcrossing to the next downcrossing of the same level, or the maximum in intervals of fixed length following the upcrossing, i.e.

$$\sup_{t\in I} \xi(t_k + t).$$

For a proof of Proposition 10.2 as well as generalizations and examples, see Lindgren (1977).

In the introduction we suggested the interpretation that the Palm probability $P_0^u\{\eta_\nu(t) \le y\}$ gives the distribution of $\{\xi(t)\}$ at time t later than the ν-th upcrossing after an arbitrary upcrossing. For this to make intuitive sense, the P_0^u-distribution of $\{\eta_\nu(t)\}$ should not depend on ν. The following theorem makes this precise.

<u>THEOREM 10.3</u> The sequence of marks $\{\eta_0(t)\}$, $\{\eta_1(t)\}$, ... is stationary under the Palm distribution P_0^u, in the sense that the joint finite-dimensional P_0^u-distribution of

$$\eta_{k+k_1}(s_j), \ldots, \eta_{k+k_r}(s_j), \; j = 1, \ldots, n$$

is independent of k. In particular, for a fixed t, all the $\eta_j(t)$, $j = 0, 1, \ldots,$ form a stationary real sequence, and hence have the same P_0^u-distribution, whereas they are non-identically distributed under P.

<u>PROOF</u> We only show that, under P_0^u, the distribution of $\eta_j(t)$ is the same as that of $\eta_{j-1}(t)$. A full proof is only notationally more complicated.

Put $\mu = E(N_u(0,1)) = E(\#\{t_k \in (0,1)\})$. Then

$$P_0^u\{\eta_j(t) \le y\} = \mu^{-1}E(\#\{t_k \in (0,1);\ \xi(t_{k+j} + t) \le y\})$$

$$= (\mu T)^{-1}E(\#\{t_k \in (0,T);\ \xi(t_{k+j} + t) \le y\}). \tag{10.6}$$

Similarly,

$$P_0^u\{\eta_{j-1}(t) \le y\} = (\mu T)^{-1}E(\#\{t_k \in (0,T);\ \xi(t_{k+j-1} + t) \le y\}). \tag{10.7}$$

Now take a pair of adjacent points t_k and t_{k+1}. We see that t_k is counted in (10.6) if and only if

$$t_k \in (0,T) \quad \text{and} \quad \xi(t_{k+j}+t) \le y$$

while t_{k+1} contributes to (10.7) if and only if

$$t_{k+1} \in (0,T) \quad \text{and} \quad \xi(t_{k+1+j-1}+t) \le y,$$

i.e. if and only if

$$t_{k+1} \in (0,T) \quad \text{and} \quad \xi(t_{k+j}+t) \le y.$$

Hence the numbers in (10.6) and (10.7) differ at most by +1 or -1 so that

$$|P_0^u\{\eta_j(t) \le y\} - P_0^u\{\eta_{j-1}(t) \le y\}| \le \frac{1}{\mu T} \to 0 \quad \text{as} \quad T \to \infty. \qquad \square$$

The Slepian model process

We will devote the rest of this chapter to a study in more detail of the properties of any of the marks under the Palm distribution, in particular as the level u gets high. In view of Theorem 10.3 all $\{\eta_k(t)\}$ have the same P_0^u-distribution and we pick $\{\eta_0(t)\}$ as a typical representative.

Our tool will be an explicit representation of the P_0^u-distribution of $\eta_0(t)$ in terms of a simple process, originally introduced by D. Slepian (1962) and therefore in this work termed a *Slepian model process*. The following theorem uses the definition of Palm distributions and forms the basis for the Slepian representation.

THEOREM 10.4 Let $\mu = E(N_u(1)) = \frac{1}{2\pi}\sqrt{\lambda_2}\, e^{-u^2/2}$. Then for $t \neq 0$,

$$(10.8) \quad P_0^u\{\eta_0(t) \le y\} = \int_{x=-\infty}^{y} \left\{\mu^{-1} \int_{z=0}^{\infty} z\, p(u,z)\, p(x|u,z)\, dz\right\} dx,$$

where $p(u,z)$ is the joint density of $\xi(0)$ and its derivative $\xi'(0)$, and $p(x|u,z)$ is the conditional density of $\xi(t)$ given $\xi(0) = u$,

$\xi'(0) = z$. Thus the P_0^u-distribution of $\eta_0(t)$ is absolutely continuous, with density

$$\mu^{-1} \int_{z=0}^{\infty} z\, p(u,z)\, p(x|u,z)\, dz.$$

The n-dimensional P_0^u-distribution of $\eta_0(s_1), \ldots, \eta_0(s_n)$ is obtained by replacing $p(x|u,z)$ by $p(x_1, \ldots, x_n|u,z)$, the conditional density of $\xi(s_1), \ldots, \xi(s_n)$ given $\xi(0) = u$, $\xi'(0) = z$.

<u>PROOF</u> The one-dimensional form (10.8) is a direct consequence of Lemmas 6.9(iii) and 6.10, since we have assumed that $\xi(0)$ and $\xi(t)$ have a non-singular distribution. We can take $\zeta(s) = \xi(s)$, $\zeta'(s) = \xi'(s)$, $\eta(s) = \xi(s+t)$ and, in the same way as in the proof of Lemma 6.11, check that

$$P\{\zeta(t) = u,\ \eta(t) = v \text{ for some } t \in (0,1)\} = 0$$

so that

$$E(\tilde{N}_u(1)) = \int_{z=0}^{\infty} z\, p(u,z)\, P\{\xi(t) \le y | \xi(0) = u,\ \xi'(0) = z\} dz$$

$$= \int_{x=-\infty}^{y} \int_{z=0}^{\infty} z\, p(u,z)\, p(x|u,z)\, dz\, dx.$$

The multivariate version is proved in an analogous way. □

Theorem 10.4 states that the joint density of $\eta_0(s_1), \ldots, \eta_0(s_n)$ under P_0^u is given by

$$(10.9) \quad \mu^{-1} \int_{z=0}^{\infty} z\, p(u,z)\, p(x_1, \ldots, x_n|u,z)\, dz,$$

where $p(x_1, \ldots, x_n|u,z)$ is the conditional density of $\xi(s_1), \ldots, \xi(s_n)$ given $\xi(0) = u$, $\xi'(0) = z$. We shall now evaluate (10.9) in order to obtain the Slepian model process.

With $\mu = \frac{1}{2\pi} \lambda_2^{1/2} e^{-u^2/2}$ and using the fact that $\xi(0)$ and $\xi'(0)$ are independent and normal with $E(\xi'(0)) = 0$, $E(\xi'(0)^2) = \lambda_2$ we have that

$$p(u,z) = \frac{u}{\lambda_2} e^{-z^2/2\lambda_2}$$

and we can write (10.9) in the form

$$(10.10) \quad \int_{z=0}^{\infty} \frac{z}{\lambda_2} e^{-z^2/2\lambda_2} p(x_1, \ldots, x_n | u,z) dz.$$

The covariance matrix of $\xi(0)$, $\xi'(0)$, $\xi(s_1)$, ..., $\xi(s_n)$ is

$$\begin{bmatrix} 1 & 0 & r(s_1) & \ldots & r(s_n) \\ 0 & \lambda_2 & -r'(s_1) & \ldots & -r'(s_n) \\ r(s_1) & -r'(s_1) & 1 & \ldots & r(s_n-s_1) \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ r(s_n) & -r'(s_n) & r(s_1-s_n) & \ldots & 1 \end{bmatrix}$$

From standard properties of conditional normal densities – see Rao (1973), p. 522 – it follows that $p(x_1, \ldots, x_n | u,z)$ is an n-variate normal density and that

$$(10.11) \quad E(\xi(s_i) | \xi(0) = u,\ \xi'(0) = z) = ur(s_i) - zr'(s_i)/\lambda_2$$

and

$$(10.12) \quad Cov(\xi(s_i),\ \xi(s_j) | \xi(0) = u,\ \xi'(0) = z)$$

$$= r(s_i - s_j) - r(s_i)r(s_j) - r'(s_i)r'(s_j)/\lambda_2.$$

The density (10.10) is therefore a mixture of n-variate normal densities, all with the same covariances (10.12), but with different means (10.11), and mixed in proportion to the Rayleigh density

$$(10.13) \quad \frac{z}{\lambda_2} e^{-z^2/2\lambda_2}, \quad (z > 0).$$

Now we are ready to introduce the Slepian model process. Let ζ be a Rayleigh distributed random variable, with density (10.13) and let

$\{\kappa(t),\ t \in R\}$ be a non-stationary normal process, independent of ζ with zero mean, and with the covariance function

$$r_\kappa(s,t) = \mathrm{Cov}(\kappa(s),\kappa(t)) = r(s-t) - r(s)r(t) - r'(s)r'(t)/\lambda_2 .$$

That this actually is a covariance function follows from (10.12).

DEFINITION 10.5 The process

(10.14) $\xi_u(t) = ur(t) - \zeta r'(t)/\lambda_2 + \kappa(t)$

is called a Slepian model process. □

Obviously, conditional on $\zeta = z$, the process (10.14) is normal with mean and covariances given by the right hand side of (10.11) and (10.12) respectively, and so its finite-dimensional distributions are given by the densities (10.10).

THEOREM 10.6 **The finite-dimensional Palm distributions of the mark $\{\eta_0(t)\}$ and thus, by Theorem 10.3, of all marks $\{\eta_k(t)\}$, are equal to the finite-dimensional distributions of the Slepian model process**

$$\xi_u(t) = ur(t) - \zeta r'(t)/\lambda_2 + \kappa(t)$$

i.e. $\quad P_0^u\{\eta_0(s_j) \in B_j,\ j=1,\dots,n\} = P\{\xi_u(s_j) \in B_j,\ j=1,\dots,n\}$. □

One should note that the height of the level u enters in $\xi_u(t)$ only via the function $ur(t)$, while the distributions of ζ and $\kappa(t)$ are the same for all u. This makes it possible to obtain the Palm distributions for the marks at crossings of any level u by introducing just one random variable ζ and one stochastic process $\{\kappa(t)\}$. In the sequel we will use the fact that u enters only through the term $ur(t)$ to derive convergence theorems for $\xi_u(t)$ as $u \to \infty$. These are then translated into the distributional convergence under the Palm distribution P_0^u, by Theorem 10.6, and thus of the limiting empirical distributions by Proposition 10.2.

As noted on p. the same reasoning applies to the limiting empirical distributions of certain other functionals. In particular, this applies to maxima, and therefore it is of interest to examine the asymptotic properties of maxima in the Slepian model process.

Some simple facts about the model process $\{\xi_u(t)\}$ should be mentioned here. From the form of the covariance function r_κ one may show that $\{\kappa(t)\}$ is continuously differentiable with

$$E(\kappa(t)) = E(\kappa'(t)) = 0$$

$$E(\kappa(0)^2) = E(\kappa'(0)^2) = 0$$

so that $P\{\kappa(0) = \kappa'(0) = 0\} = 1$. Since $\lambda_2 = -r''(0)$ one has

$$\xi_u'(0) = ur'(0) - r''(0)/\lambda_2 + \kappa'(0) = \zeta$$

so that ζ is simply the derivative of $\xi_u(t)$ at zero. From Theorem 10.6 this immediately translates into a distributional result for the derivative at upcrossings.

<u>COROLLARY 10.7</u> The Palm distribution of the mean square derivative $\eta_k'(0)$ *of a mark at a u-upcrossing does not depend on* u, *and it has* the Rayleigh density (10.13). □

<u>PROOF</u> Equality of finite dimensional distributions of course implies equality of the distributions of the difference ratios $(\eta_k(h) - \eta_k(0))/h$ and $(\xi_u(h) - \xi_u(0))/h$ and hence of their mean square limits. □

The value of ζ determines the slope of $\xi_u(t)$ at 0. For large t-values the dominant term in $\xi_u(t)$ will be $\kappa(t)$, if $r(t)$ and $r'(t)$ $\to 0$ as $t \to \infty$. (A sufficient condition for this is that the process $\{\xi(t)\}$ has a spectral density, in which case it is also ergodic.) Then $r(\tau + s, \tau + t) \to r(s - t)$ as $\tau \to \infty$ so that $\xi_u(t)$ for large t has asymptotically the same covariance structure as the unconditioned process $\xi(t)$, simply reflecting the fact that the influence of the upcrossing vanishes.

Excursions above a high level

We now turn to the asymptotic form of the marks at high level crossings. The length and height of an excursion over the high level u will turn out to be of the order u^{-1}, so we normalize the model process $\xi_u(t)$ by expanding the scale by a factor u. Before proceeding to details we give a heuristic argument motivating the precise results to be obtained, by introducing the expansion

$$
\begin{aligned}
r(\tfrac{t}{u}) &= 1 - \frac{\lambda_2 t^2}{2u^2}(1+o(1)) \\
r'(\tfrac{t}{u}) &= -\frac{\lambda_2 t}{u}(1+o(1))
\end{aligned}
\qquad (10.15)
$$

as $t/u \to 0$, which follow from (10.1), and by noting that $\kappa(t/u) = o(t/u)$ as $t/u \to 0$. Inserting this into $\xi_u(t)$ and omitting all o-terms we obtain

$$(10.16)\quad \xi_u(t/u) \sim u(1-\lambda_2 t^2/2u^2) + \zeta t/u = u + \frac{1}{u}\{\zeta t - \lambda_2 t^2/2\}$$

as $u \to \infty$ and t is fixed.

The polynomial $\zeta t - \lambda_2 t^2/2$ in (10.16) has its maximum for $t = \zeta/\lambda_2$ with a maximum value of $\zeta^2/2\lambda_2$ and therefore we might expect that $\xi_u(t)$ has a maximum of the order $u + \frac{1}{u}\zeta^2/2\lambda_2$. Hence the probability that the maximum exceeds $u + v/u$ should be approximately

$$P\{\zeta^2/2\lambda_2 > v\} = \int_{\sqrt{2\lambda_2 v}}^{\infty} \frac{z}{\lambda_2} e^{-z^2/2\lambda_2}\, dz = e^{-v}.$$

The following theorem justifies the approximations made above.

THEOREM 10.8 Suppose r satisfies (10.2) and $r(t) \to 0$ as $t \to \infty$. Then for each $\tau > 0$,

$$P\{\sup_{0\le t\le\tau} \xi_u(t) > u + \frac{v}{u}\} \to e^{-v} \quad \text{as} \quad u \to \infty,$$

i.e. the normalized height of the excursion of $\xi_u(t)$ over u is asymptotically exponential.

PROOF We first prove that the maximum of $\xi_u(t)$ occurs near zero. Choose a function $\delta(u) \to \infty$ as $u \to \infty$ such that $\delta(u)/u \to 0$ and $\delta^2(u)/u \to \infty$. Then

(10.17) $P\{\sup_{\delta(u)/u \le t \le \tau} \xi_u(t) > u\} \to 0,$

since the probability is at most

$$P\{\sup_{\delta(u)/u \le t \le \tau} (\xi_u(t) - ur(t)) + \sup_{\delta(u)/u \le t \le \tau} ur(t) > u\}$$

$$\le P\{\sup_{0 \le t \le \tau} (-\zeta r'(t)/\lambda_2 + \kappa(t)) > u(1 - \sup_{\delta(u)/u \le t \le \tau} r(t))\}.$$

Here

$$\sup_{0 \le t \le \tau} (-\zeta r'(t)/\lambda_2 + \kappa(t))$$

is a proper (i.e. finite valued) random variable, and (since $1 - r(t) = \lambda_2 t^2/2 + o(t^2)$ as $t \to 0$, and the joint distribution of $\xi(0)$ and $\xi(t)$ is nonsingular for all t so that $r(t) < 1$ for $t \neq 0$) $1 - r(t) \ge Kt^2$ for $0 \le t \le \tau$, some K depending on τ, so that

$$u(1 - \sup_{\delta(u)/u \le t \le \tau} r(t)) \ge Ku \frac{\delta^2(u)}{u^2} \to \infty$$

which implies (10.17).

In view of (10.17) we now need only show that

$$P\{\sup_{0 \le t \le \delta(u)/u} \xi_u(t) > u + \frac{v}{u}\} = P\{\sup_{0 \le t \le \delta(u)} u(\xi_u(t/u) - u) > v\} \to e^{-v}$$

for $v > 0$. By (10.15)

(10.18) $u(\xi_u(t/u) - u) = -u^2(1 - r(t/u)) - \zeta ur'(t/u)/\lambda_2 + u\kappa(t/u) =$

$$= -\lambda_2 t^2/2 \cdot (1 + o(1)) + \zeta t(1 + o(1)) + t \frac{\kappa(t/u)}{t/u}$$

uniformly for $0 \le t \le \delta(u)$ as $u \to \infty$. Since $\{\kappa(t)\}$ has a.s. continuously differentiable sample paths with $\kappa'(0) = 0$, and $\delta(u)/u \to 0$,

$$\sup_{0 \le t \le \delta(u)} \frac{\kappa(t/u)}{t/u} \to 0 \quad \text{(a.s.) as } u \to \infty.$$

This implies that the maximum of $u(\xi_u(t/u)-u)$ is asymptotically determined by the maximum of $-\lambda_2 t^2/2+\zeta t$ and that

$$\lim_{u\to\infty} P\{\sup_{0\le t\le\tau} \xi_u(t) > u+\frac{v}{u}\} = P\{\sup_t(-\lambda_2 t^2/2+\zeta t) > v\} = P\{\zeta^2/2\lambda_2 > v\}$$

$$= e^{-v}$$

as was to be shown. □

As mentioned above, distributional results and limits for the model process $\{\xi_u(t)\}$ carry over to similar results and limits for marks $\eta_k(t) = \xi(t_k+t)$, i.e. for the ergodic behaviour of the original process $\{\xi(t)\}$ after t_k.

In particular Theorem 10.8 has the corollary that the limiting empirical distribution of the normalized maxima after upcrossings of a level u is approximately exponential for large values of u, i.e.

$$\lim_{T\to\infty} \frac{\#\{t_k \in (0,T);\ \sup_{0\le t\le\tau} \xi(t_k+t) > u+\frac{v}{u}\}}{\#\{t_k \in (0,T)\}} \to e^{-v}$$

as $u\to\infty$, a.s.

This clarifies the observation in the beginning of this section, that an excursion over the high level u also exceeds the level $u-\frac{\log p}{u}$ with probability $e^{\log p} = p$.

It should be noted here, even if not formally proved, that the excursions emerging from different upcrossings are asymptotically independent. This explains the asymptotic independence of extinctions of crossings with increasing levels.

The following theorem follows from (10.18).

<u>THEOREM 10.9</u> Suppose r satisfies (10.2) and $r(t)\to 0$ as $t\to\infty$. Then with probability one the normalized model process

$$\tilde{\xi}_u(t) = u(\xi_u(t/u)-u)$$

tends uniformly for $|t|\le\tau$ to a parabola

$$\tilde{\xi}_\infty(t) = -\lambda_2 t^2/2+\zeta t$$

in the sense that, with probability one,

$$\sup_{|t|\le\tau} |\tilde{\xi}_u(t) - \tilde{\xi}_\infty(t)| \to 0$$

as $u \to \infty$. □

This theorem throws some light upon the discrete approximation used in the proof of the maximum and Poisson theorems in previous chapters. The choice of spacing in the discrete grid, q, appeared there to be chosen for purely technical reasons. Theorem 10.9 shows why it works. By the theorem the natural time scale for excursions over the high level u is u^{-1}, so the spacing $q = o(u^{-1})$ of the q-grid catches high maxima with an increasing number of grid points.

CHAPTER 11

MAXIMA AND MINIMA AND EXTREMAL THEORY FOR DEPENDENT PROCESSES

The main thread in the previous chapters has been the extension of extremal results for sequences of independent random variables, to dependent variables and to continuous time processes. Trivially, extremes in mutually independent processes are also independent, and we shall see that this holds asymptotically for normal processes even when they are highly correlated.

However, we shall first consider the joint distribution of maxima and minima in one process. (Since minima for $\xi(t)$ are maxima for $-\xi(t)$, this can in fact be regarded as a special case of dependence, namely between $\{\xi(t)\}$ and $\{-\xi(t)\}$.)

Maximum and minimum

Suppose $\xi_1, \xi_2, \ldots$ is a stationary sequence of independent random variables with

$$P\{\xi_1 > u_n\} \sim \frac{\tau}{n}, \quad P\{\xi_1 < -v_n\} \sim \frac{\sigma}{n}$$

as $n \to \infty$, so that

$$P\{\max_{1\le i\le n} \xi_i \le u_n\} \to e^{-\tau}, \quad P\{\min_{1\le i\le n} \xi_i \ge -v_n\} \to e^{-\sigma}.$$

Then clearly

$$\begin{aligned} P\{-v_n \le \min_{1\le i\le n} \xi_i \le \max_{1\le i\le n} \xi_i \le u_n\} &= P\{-v_n \le \xi_1 \le u_n\}^n \\ &= (1 - P\{\xi_1 > u_n\} - P\{\xi_1 < -v_n\})^n \\ &= (1 - \frac{\tau+\sigma}{n} + o(n^{-1}))^n \to e^{-\tau}e^{-\sigma}, \end{aligned}$$

i.e. maximum and minimum are asymptotically independent.

For a standardized stationary normal process $\{\xi(t)\}$ and $\{-\xi(t)\}$ have the same distribution. Writing

$$m(T) = \inf\{\xi(s);\ 0 \le s \le T\},$$

clearly $m(T) = -\sup\{-\xi(s);\ 0 \le s \le T\}$, and hence $m(T)$ has the same asymptotic behaviour as $-M(T)$. If $\{\xi(t)\}$ satisfies the hypotheses of Theorem 7.6,

$$P\{m(T) \ge -v\} \to e^{-\sigma}$$

as $T, v \to \infty$ and $T\nu \to \sigma > 0$ for

$$\nu = \frac{1}{2\pi} \lambda_2^{1/2} e^{-v^2/2}.$$

It follows, under the hypotheses of Theorem 7.8, that

$$P\{a_T(m(T) - b_T) \le x\} \to 1 - \exp(-e^x)$$

with the same normalizations as for maxima, i.e.

$$a_T = (2 \log T)^{1/2}$$

$$b_T = (2 \log T)^{1/2} + \frac{\log(\lambda_2^{1/2}/2\pi)}{(2 \log T)^{1/2}}.$$

It was shown by Berman (1971 a) that under no further assumptions $m(T)$ and $M(T)$ are asymptotically independent as in the case of independent sequences studied above. We shall now prove this result.

With the same notation and technique as in Chapter 7, we let N_u and $N_u^{(q)}$ be the number of u-upcrossings by the process $\{\xi(t);\ 0 \le t \le h\}$ and the sequence $\{\xi(jq);\ 0 \le jq \le h\}$, and define similarly D_{-v} and $D_{-v}^{(q)}$ to be the number of *downcrossings* of the level $-v$. We first observe that if, for $h = 1$,

$$\mu = E(N_u) = \frac{1}{2\pi} \lambda_2^{1/2} e^{-u^2/2},$$

$$\nu = E(D_{-v}) = \frac{1}{2\pi} \lambda_2^{1/2} e^{-v^2/2},$$

and $u, v \to \infty$ so that $T\mu \to \tau > 0$ and $T\nu \to \sigma > 0$, then we have $u \sim v$ and

$$(11.1)\quad u - v \sim \frac{\log \sigma/\tau}{u},$$

cf. (9.7). In particular, this implies that if $q \to 0$ so that $uq \to 0$ then also $vq \to 0$.

The following lemma contains the discrete approximation and separation of maxima. As in Chapter 7, split the increasing interval $[0,T]$ into $n = [T/h]$ pieces, each divided into two, I_k and I_k^*, of length $h - \varepsilon$ and ε, respectively. Write $m(I)$ for the minimum of $\xi(t)$ over the interval I.

<u>LEMMA 11.1</u> If (7.1) holds then, as $u, v \to \infty$ and $uq, vq \to 0$,

(i) $$P\{m(I) \ge -v,\ M(I) \le u\}$$
$$= P\{\min_{jq \in I} \xi(jq) \ge -v,\ \max_{jq \in I} \xi(jq) \le u\} + o(\mu + \nu),$$

where $o(\mu + \nu)$ is uniform in all intervals of fixed length h,

(ii) $$\limsup_{T \to \infty} \left| P\{-v \le m(\bigcup_{k=1}^{n} I_k) \le M(\bigcup_{k=1}^{n} I_k) \le u\} -\right.$$
$$\left. - P\{-v \le m(nh) \le M(nh) \le u\} \right| \le \frac{\mu + \nu}{h}\,\varepsilon,$$

(iii) $$P\{-v \le \xi(jq) \le u,\ jq \in \bigcup_{k=1}^{n} I_k\} -$$
$$- P\{-v \le m(\bigcup_{k=1}^{n} I_k) \le M(\bigcup_{k=1}^{n} I_k) \le u\} \to 0.$$

<u>PROOF</u> (i) By Lemma 7.3 (i) applied to $\{\xi(t)\}$ and $\{-\xi(t)\}$,

$$0 \le P\{m(I) \ge -v,\ M(I) \le u\} - P\{\min_{jq \in I} \xi(jq) \ge -v,\ \max_{jq \in I} \xi(jq) \le u\}$$

$$\le E(D_{-v} - D_{-v}^{(q)}) + E(N_u - N_u^{(q)}) + P\{\xi(0) < -v\} + P\{\xi(0) > u\}$$

$$= o(\mu + \nu).$$

Parts (ii) and (iii) follow as in Lemma 7.4 and the details are not repeated here. □

The following two lemmas give the asymptotic independence of both maxima and minima over the separate I_k-intervals.

LEMMA [illegible].2 Suppose [illegible], ..., [illegible] are standard normal variables with [illegible] variance matrix [illegible], and [illegible], ..., [illegible] [illegible], with [illegible] matrix [illegible], and let [illegible] max [illegible]. Further, let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be vectors of real numbers and write [illegible] $x_1, \ldots, x_n, y_1, \ldots, y_n$. Then

[illegible].2. [illegible] for [illegible], ..., [illegible]

$$[\text{illegible}]\; e^{-x^2}\; [\text{illegible}].$$

PROOF [illegible] requires only a minor variation of the proof of Lemma [illegible].2. With the notation of that proof, it follows as on p. 46-47 that the lefthand-side of [illegible].2 equals

$$[\text{illegible}] \frac{\partial^2 f_n}{\partial y_i \partial y_j}\, dy\, dh.$$

By performing the integrations in y_i and y_j we obtain four terms, containing the integrands $f_n(y_i = x_i, y_j = x_j)$, $-f_n(y_i = x_i, y_j = -x_j)$, $-f_n(y_i = -x_i, y_j = x_j)$, and $f_n(y_i = -x_i, y_j = -x_j)$, respectively. Each of these terms can be estimated as on p. 47, and the lemma follows. □

LEMMA [illegible] Suppose [illegible] as [illegible], and that $T_1 \to$ [illegible], $T_2 \to$ [illegible] as $T \to \infty$, and that, if [illegible] $\to 0$ sufficiently slowly

[illegible] $$[\text{illegible}]\; e^{-w^2}\; [\text{illegible}] \to 0$$

for every [illegible] [illegible], which in particular holds if [illegible] $\to 0$ as [illegible]. Then

(i) $$[\text{illegible}] \sum_{j=1}^{n} [\text{illegible}] \to 0,$$

(ii) $$\limsup_{T \to \infty} [\text{illegible}]\; \frac{2[\text{illegible}]}{n}.$$

PROOF Part (i) follows from Lemma 11.2 in the same way as Lemma 7.5 (i) follows from Lemma 7.2. As for part (ii), note that

$$0 \le P(-v < m(I_j) \le M(I_j) \le u) - P(-v < m(h) \le M(h) \le u)$$

$$\le P(m(I_j^*) \le -v) + P(M(I_j^*) > u) = o(\lambda + \mu)$$

by Lemma 7.2 (i). It now follows from Lemma 11.1 (i) and Lemma 7.2 (i) that, for T sufficiently large,

$$0 \le P_j - P \le 2(\lambda + \mu)\varepsilon$$

where $P_j = P(-v < \xi(kq) \le u,\ kq \in I_j)$, $P = P(-v < m(h) \le M(h) \le u)$. Hence

$$0 \le \prod_{j=1}^{n} P_j - P^n \le 2n(\lambda + \mu)\varepsilon \sim \frac{2(\lambda + \mu)}{h}\varepsilon$$

as in Lemma 7.5 (ii). □

The final essential step in the proof of asymptotic independence of $m(T)$ and $M(T)$ is to show that an interval of fixed length h does not contain both large positive and large negative values of $\xi(t)$.

LEMMA 11.4 If (7.1) holds, there is an $h_0 > 0$ such that as $u, v \to \infty$,

$$P(M(h) > u,\ m(h) < -v) = o(\lambda + \mu)$$

for all $h < h_0$.

PROOF By Lemma 11.1 (i) and stationarity it is enough to prove that

$$P\left(\max_{0 \le jq \le h} \xi(jq) > u,\ \min_{0 \le jq \le h} \xi(jq) < -v\right) = o(\lambda + \mu)$$

for some q satisfying $uq, vq \to 0$. By stationarity, this probability is bounded by

(11.4) $$\frac{h}{q} P\left(\xi(0) > u,\ \min_{-h \le jq \le h} \xi(jq) < -v\right) \le \frac{h}{q} \sum_{-h \le jq \le h} P(\xi(0) > u,\ \xi(jq) < -v).$$

Here, for $jq \ne 0$,

$$P\{\xi(0) > u,\ \xi(jq) < -v\} = \int_u^\infty \phi(x) P\{\xi(jq) < -v \mid \xi(0) = x\} dx$$

$$(11.5) \qquad = \int_u^\infty \phi(x) \Phi\left(\frac{-v - xr(jq)}{\sqrt{1 - r^2(jq)}}\right) dx,$$

since, conditional on $\xi(0) = x$, $\xi(jq)$ is normal with mean $xr(jq)$ and variance $1 - r^2(jq)$. Now, choose $h_0 > 0$ such that $0 < r(t) < 1$ for $0 < |t| \le h_0$, which is possible by (7.1). If $0 < jq \le h < h_0$ then

$$(-v - xr(jq))/\sqrt{1 - r^2(jq)} \le -v,$$

for $u, v > 0$, so that (11.5) is bounded by

$$\int_u^\infty \phi(x)\Phi(-v)dx = (1 - \Phi(u))(1 - \Phi(v)) \sim \frac{\phi(u)\phi(v)}{uv}.$$

Together with (11.4) and (11.5) this shows that

$$0 \le P\{\max_{0 \le jq \le h} \xi(jq) > u,\ \min_{0 \le jq \le h} \xi(jq) < -v\} \le \frac{h^2}{q^2} \cdot \frac{\phi(u)\phi(v)}{uv}$$

$$= h^2 \cdot \frac{\phi(u)\phi(v)}{(\mu + \nu)} \cdot \frac{1}{quqv} \quad (\mu + \nu) = o(1)(\mu + \nu),$$

since $\phi(u)\phi(v)/(\mu + \nu) = O(\mu\nu/(\mu + \nu)) \to 0$, and q can be chosen to make $quqv \to 0$ arbitrarily slowly. □

<u>THEOREM 11.3</u> Let $u = u_T \to \infty$ and $v = v_T \to \infty$ as $T \to \infty$, in such a way that

$$T\mu = \frac{T}{2\pi} \lambda_2^{1/2} e^{-u^2/2} \to \tau > 0,$$

$$T\nu = \frac{T}{2\pi} \lambda_2^{1/2} e^{-v^2/2} \to \sigma > 0.$$

Suppose $r(t)$ satisfies (7.1) and either (7.2) or the weaker (11.3). Then

$$P\{-v < m(T) \le M(T) \le u\} \to e^{-\sigma - \tau} \quad \text{as} \quad T \to \infty,$$

and hence

$$P\{a_T(m(T) - b_T) \le x,\ a_T(M(T) - b_T) \le y\} \to (1 - \exp(-e^{x}))\exp(-e^{-y})$$

with $a_T = (2 \log T)^{1/2}$, $b_T = (2 \log T)^{1/2} + \log(\lambda_2^{1/2}/2\pi)/(2 \log T)^{1/2}$.

PROOF By Lemma 11.1 (ii) and (iii), and Lemma 11.3 (i) and (ii) we have

$$\limsup_{T \to \infty} \left| P\{-v < m(nh) \le M(nh) \le u\} - P^n\{-v < m(h) \le M(h) \le u\} \right| \le \frac{3(\sigma + \tau)}{h} \varepsilon ,$$

for arbitrary $\varepsilon > 0$, and hence

$$(11.6) \quad P\{-v < m(nh) \le M(nh) \le u\} - P^n\{-v < m(h) \le M(h) \le u\} \to 0$$

as $n \to \infty$. Furthermore, by Lemma 11.4,

$$P^n\{-v < m(h) \le M(h) \le u\} = (1 - P\{m(h) \le -v\} - P\{M(h) > u\} + o(\nu + \mu))^n .$$

Arguing as in the proof of Theorem 7.6, the result now follows from Lemma 7.3, using $n(\nu + \mu) \sim \frac{T}{h}(\nu + \mu) \to \frac{\sigma + \tau}{h}$. □

This theorem of course has similar ramifications as the maximum theorem, Theorem 7.6, but before stating them we give a simple corollary about the absolute maximum of $\xi(t)$.

COROLLARY 11.6 If $u \to \infty$, $T = \tau/\mu$, then

$$P\{\sup_{0 \le t \le T} |\xi(t)| \le u\} \to e^{-2\tau},$$

and furthermore

$$P\{a_T(\sup_{0 \le t \le T} |\xi(t)| - b_T) \le x + \log 2\} \to \exp(-e^{-x}) . \quad □$$

As for the maximum alone, it is now easy to prove asymptotic independence of maxima and minima over several disjoint intervals with lengths proportional to T. As a consequence one has a Poisson convergence theorem for the two point processes of upcrossings of u and downcrossings of $-v$, the limiting Poisson processes being independent. Furthermore, the point process of downcrossings of several low levels converges to a point process with Poisson components obtained by successive binomial thinning, as in Theorem 8.6, and these downcrossings processes are asymptotically independent of the upcrossings processes. Of course, the entire point process of local minima, considered in Theorem 9.4, is also asymptotically independent of the point process of normalized local minima.

Extreme values and crossings for dependent processes

One remarkable feature of dependent normal processes is that, regardless of how high the correlation – short of perfect correlation – the number of high level crossings in the different processes are asymptotically independent, as shown in Lindgren (1974). This shall now be proved, again by means of the important Lemma 3.2.

Let $\{\xi_1(t)\}, \ldots, \{\xi_p(t)\}$ be jointly normal processes with zero means, variances one and covariance functions $r_k(\tau) = \mathrm{Cov}(\xi_k(t), \xi_k(t+\tau))$. We shall assume that they are jointly stationary, i.e. $\mathrm{Cov}(\xi_k(t), \xi_\ell(t+\tau))$ does not depend on τ, and we write

$$r_{k\ell}(\tau) = \mathrm{Cov}(\xi_k(t), \xi_\ell(t+\tau))$$

for the *cross-covariance* function. Suppose further that each r_k satisfies (7.1), possibly with different λ_k's, i.e.

$$r_k(t) = 1 - \lambda_{2k}\, t^2/2 + o(t^2), \quad t \to 0, \tag{11.7}$$

and that

$$\begin{aligned} &r_k(t) \log t \to 0, \\ &r_{k\ell}(t) \log t \to 0, \quad \text{as } t \to \infty, \end{aligned} \tag{11.8}$$

for $1 \le k, \ell \le p$. To exclude the possibility that $\xi_k(t) \equiv \pm\xi_\ell(t+t_0)$ for some $k \ne \ell$, and some choice of t_0 and $+$ or $-$, we assume that

$$\max_{k \ne \ell} \sup_t |r_{k\ell}(t)| < 1. \tag{11.9}$$

However, we note here that if $\inf_t r_{k\ell}(t) = -1$ for some $k \ne \ell$, there is a t_0 such that $r_{k\ell}(t_0) = -1$, which means that $\xi_k(t) \equiv -\xi_\ell(t+t_0)$. A maximum in $\xi_\ell(t)$ is therefore a minimum in $\xi_k(t)$, and as was shown in the first section of this chapter, maxima and minima are asymptotically independent. In fact, with some increase in the complexity of proof, condition (11.9) can be relaxed to $\max_{k \neq \ell} \sup_t r_{k\ell}(t) < 1$.

Define

$$M_k(T) = \sup\{\xi_k(t);\ 0 \leq t \leq T\}, \quad k = 1, \ldots, p$$

and let $u_k = u_k(T)$ be levels such that

$$T\mu_k = \frac{T}{2\pi}\sqrt{\lambda_{2k}}\, e^{-u_k^2/2} \to \tau_k > 0$$

as $T \to \infty$. Write $\mu = \min\{\mu_1, \ldots, \mu_p\}$.

To prove asymptotic independence of the $M_k(T)$ we approximate by the maxima over separated intervals I_j, $j = 1, \ldots, n$, with $n = [T/h]$ for h fixed, and then replace the continuous maxima by the maxima of the sampled processes to obtain asymptotical independence of maxima over *different* intervals. We will only briefly point out the changes which have to be made in previous arguments. The main new argument to be used here concerns the maxima of $\xi_k(t)$, $k = 1, \ldots, p$ over one fixed interval, I, say.

We first state the asymptotic independence of maxima over disjoint intervals.

<u>LEMMA 11.7</u> If r_k, $r_{k\ell}$ satisfy (11.7)-(11.9) for $1 \leq k, \ell \leq p$, and if $T\mu_k \to \tau_k > 0$, then for $h > 0$ and $n = [T/h]$,

$$P\{M_k(nh) \leq u_k,\ k = 1, \ldots, p\} - P^n\{M_k(h) \leq u_k,\ k = 1, \ldots, p\} \to 0.$$

<u>PROOF</u> This corresponds to (11.6) in the proof of Theorem 11.5, and is proved by similar means. It is only the relation

$$(11.10)\quad P\{\xi_k(jq) \leq u_k,\ jq \in \bigcup_{r=1}^{n} I_r,\ k = 1, \ldots, p\} - $$

$$- \prod_{r=1}^{n} P\{\xi_k(jq) \leq u_k,\ jq \in I_r,\ k = 1, \ldots, p\} \to 0,$$

corresponding to Lemma 11.3 (i), that has to be given a different proof.

Identifying $\xi_1, \ldots, \xi_n$ in Lemma 3.2 with $\xi_1(jq), \ldots, \xi_p(jq)$, $jq \in \bigcup_{r=1}^{n} I_r$, and $\eta_1, \ldots, \eta_n$ analogously, but with variables from different I_r-intervals independent, (3.6) gives, since $\sup r_{k\ell}(t) < 1$,

$$|P\{\xi_k(jq) \le u_k,\ jq \in \bigcup_{r=1}^{n} I_r,\ k = 1, \ldots, p\} -$$

$$- \prod_{r=1}^{n} P\{\xi_k(jq) \le u_k,\ jq \in I_r,\ k = 1, \ldots, p\}|$$

$$\le K \sum_{k=1}^{p} \sum_{i<j}^{*} |r_k((i-j)q)|\, e^{-u^2/(1+|r_k((i-j)q)|)} + \tag{11.11}$$

$$+ K \sum_{1\le k\ne\ell\le p} \sum_{i<j}^{*} |r_{k\ell}((i-j)q)|\, e^{-u^2/(1+|r_{k\ell}((i-j)q)|)}$$

where Σ^* as before indicates that the sum is taken over i, j such that iq and jq belong to different I_r. Since both $r_k(t) \log t \to 0$, $r_{k\ell}(t) \log t \to 0$ and $\sup|r_{k\ell}(t)| < 1$ we can, as in Lemma 7.1 (ii), conclude that both sums in (11.11) tend to zero. □

<u>LEMMA 11.8</u> If r_k, $r_{k\ell}$ satisfy (11.7) and (11.9), for $1 \le k, \ell \le p$, then

(i) $$P\{M_k(h) > u_k,\ M_\ell(h) > u_\ell\} = o(\mu_k + \mu_\ell) \quad \text{for } k \ne \ell,$$

and

(ii) $$P\{M_k(h) \le u_k,\ k = 1, \ldots, p\} = 1 - \sum_{k=1}^{p} P\{M_k(h) > u_k\} + o(\sum_{k=1}^{p} \mu_k).$$

<u>PROOF</u> (i) As in the proof of Lemma 11.4 it is enough to prove that, if $q \to 0$ so that $u_k q \sim u_\ell q \to 0$ sufficiently slowly,

$$P\{\max_{0\le jq\le h} \xi_k(jq) > u_k,\ \max_{0\le jq\le h} \xi_\ell(jq) > u_\ell\} = o(\mu_k + \mu_\ell).$$

Since, for $r = k, \ell$,

$$P\{\max_{0\le jq\le h} \xi_r(jq) > u_r\} = o(\mu_r)$$

it clearly suffices to prove that

$$(11.12)\quad P\{\max_{0\le jq\le h}\xi_k(jq)\le u_k,\ \max_{0\le jq\le h}\xi_\ell(jq)\le u_\ell\} -$$

$$- P\{\max_{0\le jq\le h}\xi_k(jq)\le u_k\} P\{\max_{0\le jq\le h}\xi_\ell(jq)\le u_\ell\} = o(\mu_k+\mu_\ell).$$

To estimate the difference we again use Lemma 3.2 with Λ^1 defined by r_k, r_ℓ, and $r_{k\ell}$, and Λ^0 obtained by taking $r_{k\ell}$ identically zero. Elementary calculation show that the difference in (11.12) equals

$$P\{\max_{0\le jq\le h}\xi_k(jq)\le u_k,\ \max_{0\le jq\le h}\xi_\ell(jq)\le u_\ell\} -$$

$$- P\{\max_{0\le jq\le h}\xi_k(jq)\le u_k\} P\{\max_{0\le jq\le h}\xi_\ell(jq)\le u_\ell\},$$

which by Lemma 3.2 is bounded in modulus by

$$(11.13)\quad \frac{1}{2\pi}\sum_{0\le iq,\,jq\le h}|r_{k\ell}((i-j)q)|(1-r_{k\ell}^2((i-j)q))^{-1/2}\,.$$

$$\cdot\ e^{-u^2/(1+|r_{k\ell}((i-j)q)|)}$$

with $u=\min(u_k,u_\ell)$.

Now, by (11.9), $\sup|r_{k\ell}(t)| = 1-\delta$ for some $\delta>0$, and using this, we can bound (11.13) by

$$Kh^2q^{-2}\exp(-u^2/(1+1-\delta))$$

$$= Kh^2\,\frac{\phi(u)}{\mu_k+\mu_\ell}(uq)^{-2}u^2\exp\left(-u^2\cdot\frac{\delta}{2(2-\delta)}\right)(\mu_k+\mu_\ell)$$

$$= o(\mu_k+\mu_\ell)$$

if $uq\to 0$ sufficiently slowly, since $\phi(u)/(\mu_k+\mu_\ell)$ is bounded.

(ii) This follows immediately from part (i) and the inequality

$$\sum_{k=1}^{p}P\{M_k(h)>u_k\} - \sum_{1\le k<\ell\le p}P\{M_k(h)>u_k,\ M_\ell(h)>u_\ell\}$$

$$\le P\left(\bigcup_{k=1}^{p}\{M_k(h)>u_k\}\right) \le \sum_{k=1}^{p}P\{M_k(h)>u_k\}. \qquad \square$$

Reasoning as in the proof of Theorem 7.6, and using Lemma 11.7 and Lemma 11.8 (ii) we get the following result.

<u>THEOREM 11.9</u> Let $u_k = u_k(T) \to \infty$ as $T \to \infty$, so that $T\mu_k = \frac{T}{2\pi}\sqrt{\lambda_{2k}}\exp(-u_k^2/2) \to \tau_k > 0$, $1 \le k \le p$, and suppose that $r_k(t)$ and $r_{k\ell}(t)$ satisfy (11.7)-(11.9). Then

$$P(M_k(T) \le u_k,\ k=1,\dots,p) \to \exp\Big(-\sum_{k=1}^{p} \tau_k\Big)$$

as $T \to \infty$. $\square$

Under the same conditions as in Theorem 11.9, the time-normalized point processes of u_k-upcrossings tend jointly in distribution to p independent Poisson processes with intensities τ_k. The precise formulation of the theorem, and its proof, is left to the reader.

We end this chapter with an example which gives an illustration to the extraordinary character of extremes in normal processes.

Let $\{\zeta(t)\}$ and $\{\eta(t)\}$ be independent standardized normal processes whose covariance functions r_ζ and r_η satisfy (7.1) and (7.2), let c_k, $k=1,\dots,p$, satisfying $|c_k| < 1$, and $c_k \ne c_\ell$, $k \ne \ell$, be constants, and define

$$\xi_k(t) = c_k\zeta(t) + (1-c_k^2)^{1/2}\eta(t).$$

Then the processes $\xi_k(t)$, $k=1,\dots,p$ are jointly normal and their covariance functions $r_k(t)$ and crosscovariance functions

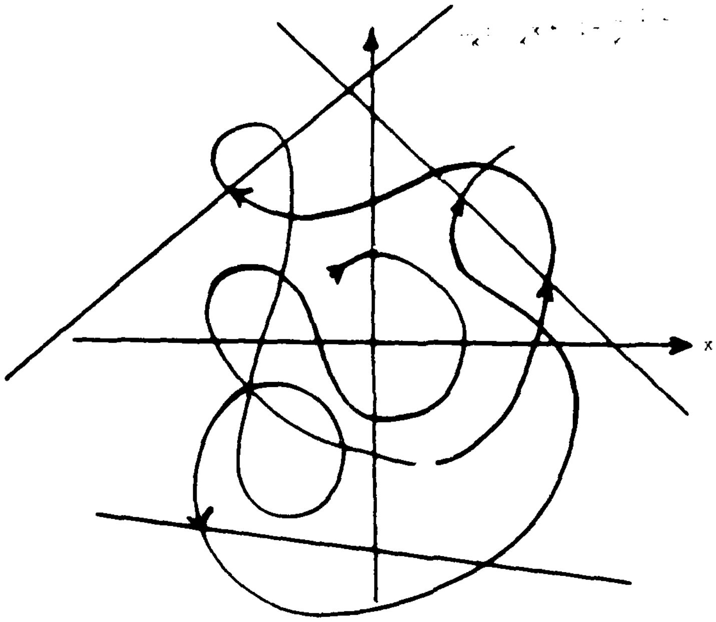
$$r_{k\ell}(t) = c_k c_\ell r_\zeta(t) + (1-c_k^2)^{1/2}(1-c_\ell^2)^{1/2} r_\eta(t)$$

satisfy (11.7)-(11.9). Thus, even though $\xi_1(t), \dots, \xi_p(t)$ are linearly dependent, their maxima are asymptotically independent.

We can illustrate this geometrically by representing $(\zeta(t), \eta(t))$ by a point moving randomly in the plane. The upcrossings of a level u_k by $\xi_k(t)$ then correspond to outcrossings of the line

$$c_k x + (1-c_k^2)^{1/2} y = u_k$$

by $(\zeta(t), \eta(t))$, as illustrated in the following diagram.



The times of these outcrossings form asymptotically independent Poisson processes, when suitably normalized, if T [illegible] $\to 0$.

CHAPTER 12

MAXIMA AND CROSSINGS OF NON-DIFFERENTIABLE NORMAL PROCESSES

The basic assumption of the previous chapters has been that the covariance function $r(\tau)$ of the stationary normal process $\xi(t)$ has an expansion

$$r(\tau) = 1 - \lambda_2\tau^2/2 + o(\tau^2) \quad \text{as} \quad \tau \to 0.$$

In this chapter we shall consider the more general class of covariance functions which have the expansion

$$(12.1) \quad r(\tau) = 1 - C|\tau|^{\alpha} + o(|\tau|^{\alpha}) \quad \text{as} \quad \tau \to 0,$$

where α is a constant, $0 < \alpha \le 2$, and C is a positive constant. This includes covariances of the form $\exp(-|\tau|^{\alpha})$, the case $\alpha = 1$ being that of the Ornstein-Uhlenbeck process. Note that we may take $\alpha = 2$ in (12.1) so that all results of this chapter are indeed true also for the regular case previously studied; cf. Theorem 12.9.

If $\alpha < 2$ we cannot expect a Poisson result to hold for the upcrossings of a high level, since r is then not twice differentiable, implying that $\lambda_2 = \infty$, and hence the mean number of crossings of any level in any interval is infinite by Rice's formula. Furthermore, it can be shown that at every point t where a crossing occurs, there are an uncountable number of crossings in every neigbourhood of t. However, it is certainly possible for the maximum $M(T)$ to have a limiting distribution. The point is that while upcrossings may be infinite in number, (12.1) is, as noted in Chapter 6, p. 8, sufficient to guarantee continuity of the sample paths of the process, and thus ensure that $M(T)$ is well defined and finite.

We shall in fact show that $a_T(M(T) - b_T)$ has a limiting double exponential distribution if the normalizing constants are

$$
(12.2) \qquad
\begin{aligned}
a_T &= (2 \log T)^{1/2} \\
b_T &= (2 \log T)^{1/2} + \frac{1}{(2 \log T)^{1/2}} \left(\frac{2-\alpha}{\alpha} \log \log T \right. \\
&\qquad \left. + \log\left(C^{1/\alpha} H_\alpha (2\pi)^{-1/2} 2^{(2-\alpha)/2\alpha}\right)\right),
\end{aligned}
$$

where H_α is a certain strictly positive constant.

This remarkable result was first obtained by Pickands (1969), although his proofs were not quite complete. Complements and extensions have been given by Berman (1971b), Qualls and Watanabe (1972), and Lindgren, de Maré, and Rootzén (1975). The method of Pickands has some particularly interesting features, in that it uses a generalized notion of upcrossings which makes it possible to obtain a Poisson type result also for $\alpha < 2$.

Specifically, Pickands considers what he terms *ε-upcrossings*. Given $\varepsilon > 0$, the function $f(t)$ is said to have an ε-upcrossing of the level u at t_0 if $f(t) \leq u$ for all $t \in (t_0 - \varepsilon, t_0)$, and, for all $\eta > 0$, $f(t) > u$ for some $t \in (t_0, t_0 + \eta)$. Clearly, this is equivalent to requiring that is has a (non-strict or strict) upcrossing there, and furthermore $f(t) \leq u$ for all t in $(t_0 - \varepsilon, t_0)$. An ε-upcrossing is always an upcrossing, while obviously an upcrossing need not be an ε-upcrossing.

Clearly, the number of ε-upcrossings in, say, a unit interval, is bounded (by $1/\varepsilon$) and hence certainly has a finite mean. Even if this mean cannot be calculated as easily as the mean number of ordinary upcrossings, its limiting form for large u has a simple relation to the extremal results for $M(T)$. In particular, as we shall see, it does not depend on the ε chosen.

The main complication in the derivation of this result, as compared to the case $\alpha = 2$, concerns the tail distribution of $M(h)$ for h fixed, which cannot be approximated with the tail distribution of the simple cosine process if $\alpha < 2$.

The proof [illegible] a number of [illegible]

to [illegible] [illegible] [illegible]

1. Find the asymptotic distribution of max [illegible] for a fixed n:

[illegible]

as [illegible] and n fixed, [illegible] a constant (Lemma 12.3).

2. Find the asymptotic distribution of max [illegible] for a fixed [illegible]:

[illegible]

[illegible]

[illegible]

as [illegible] (Lemma 12.[illegible]).

3. Approximate [illegible] max [illegible] for fixed h:

[illegible]

as [illegible] (Lemma 12.5)

and

[illegible]

as [illegible] (Lemma 12.6).

4. Find the tail distribution of sup [illegible] for a fixed h:

[illegible]

(Theorem 12.4), where

[illegible]

[illegible]

[illegible]

We start with a [illegible]

LEMMA [illegible]

[illegible]

(2.3) [illegible]

for some [illegible], then there exists a constant [illegible] depending on [illegible] such that for all x,

[illegible]

[illegible] this last term is zero.

Proof. The idea of the proof is to express all [illegible] of the form [illegible] with [illegible]

(2.4) [illegible]

and construct bounds for each of the terms in [illegible] time for [illegible]

[illegible]

and note that [illegible] is normal with [illegible]

(2.5) [illegible]

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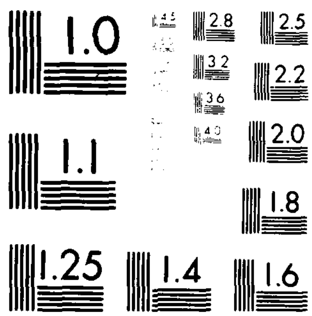

MICROCOPY RESOLUTION TEST CHART

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Now, let $\beta > 0$ and introduce the event

$$A = \bigcup_{p=0}^{\infty} \{ \max_{0 \le k \le 2^p - 1} \frac{\zeta(k,p)}{E(\zeta(k,p)^2)^{1/2}} \ge (2\beta(p+1))^{1/2} \}.$$

If $\beta > \log 2$, Boole's inequality, together with (12.5), implies

$$P(A) \le \sum_{p=0}^{\infty} \sum_{k=0}^{2^p - 1} P\{\frac{\zeta(k,p)}{E(\zeta(k,p)^2)^{1/2}} \ge (2\beta(p+1))^{1/2}\}$$

$$\le \sum_{p=0}^{\infty} 2^p e^{-\beta(p+1)} = e^{-\beta}/(1 - e^{-(\beta - \log 2)}),$$

so that if $\beta > 2 \log 2$, then

(12.6) $\quad P(A) \le 4e^{-\beta}$.

But for $\beta < 2 \log 2$ this holds trivially (since $P(A) \le 1$) and we can therefore use (12.6) for all values of $\beta \ge 0$.

Next, note that on the complementary event A^*, all the inequalities

$$|\zeta(k,p)| \le E(\zeta(k,p)^2)^{1/2} \cdot (2\beta(p+1))^{1/2}$$

hold for $p = 0, 1, \ldots; k = 0, 1, \ldots, 2^p - 1$, and that (12.3) implies that

$$E(\zeta(k,p)^2) \le C\, 2^{-(p+1)\alpha}.$$

Thus, by (12.4) we conclude that on A^*,

$$|\xi(t) - \xi(0)| \le \sum_{p=0}^{\infty} C^{1/2} 2^{-(p+1)\alpha/2} (2\beta(p+1))^{1/2} = \sqrt{\frac{\beta C}{4c_\alpha}}, \text{ say,}$$

and that consequently, by (12.6),

$$P\{ \sup_{0 \le t \le 1} |\xi(t) - \xi(0)| < \sqrt{\frac{\beta C}{4c_\alpha}} \} \le 4e^{-\beta}.$$

The conclusion of the lemma now follows by choosing $\beta = c_\alpha x^2/C$, since

$$P\{ \sup_{0 \le t \le 1} \xi(t) > x\} \le P\{ \sup_{0 \le t \le 1} |\xi(t) - \xi(0)| > x/2\} + P\{\xi(0) > x/2\}$$

$$\le 4e^{-c_\alpha x^2/C} + \frac{1}{2} e^{-x^2/8\sigma^2}. \qquad \square$$

With this result out of the way we return to the process $\{\xi(t)\}$ with covariance function $r(t)$. When considering the distribution of continuous or discrete maxima like $\sup_{0\le t\le h} \xi(t)$ and $\max_{0\le jq\le h} \xi(jq)$ for small values of h, it is natural to condition on the value of $\xi(0)$. In fact, the local behaviour (12.1) of $r(t)$ is reflected in the local variation of $\xi(t)$ around $\xi(0)$. For normal processes this involves no difficulty of definition if one considers $\xi(t)$ only at a finite number of points, say $t = t_j$, $j = 1, \ldots, n$, since conditional probabilities are then defined in terms of ratios of density functions (cf. Chapter 6 p. 16). Thus we can write, with $t_0 = 0$,

$$P\{\max_{j=0,\ldots,n} \xi(t_j) \le u\} = \int_{-\infty}^{u} \phi(x) P\{\max_{j=1,\ldots,n} \xi(t_j) \le u \mid \xi(0) = x\}\, dx,$$

where the conditional probability can be expressed by means of a (conditional) normal density function (cf. Chapter 10 p. 66). In particular the conditional probability is determined by conditional means and covariances.

For maxima over a real interval we have, e.g. with $t_j = hj2^{-n}$, $j = 0, \ldots, 2^n$,

$$P\{\sup_{0\le t\le h} \xi(t) \le u\} = \lim_{n\to\infty} P\{\max_{t_j} \xi(t_j) \le u\},$$

which by dominated convergence equals

$$\int_{-\infty}^{u} \phi(x) \lim_{n\to\infty} P\{\max_{t_j} \xi(t_j) \le u \mid \xi(0) = x\}\, dx$$

Now, if the conditional means and covariances of $\xi(t)$ given $\xi(0)$ are such that the normal process they define is continuous, we define

$$P\{\sup_{0\le t\le u} \xi(t) \le u \mid \xi(0) = x\}$$

to be the probability that, in a continuous normal process with mean $E(\xi(t) \mid \xi(0) = x)$ and covariance function $\mathrm{Cov}(\xi(s), \xi(t) \mid \xi(0) = x)$, the maximum does not exceed u. Then clearly

$$\lim_{n\to\infty} P\{\max_{t_j} \xi(t_j) \le u \mid \xi(0)=x\} = P\{\sup_{0\le t\le h} \xi(t) \le u \mid \xi(0)=x\}.$$

In this case we have, by dominated convergence,

$$P\{\sup_{0\le t\le h} \xi(t) \le u\} = \int_{-\infty}^{u} \phi(x) P\{\sup_{0\le t\le h} \xi(t) \le u \mid \xi(0)=x\}\, dx.$$

In the applications below, the conditional distributions define a continuous process, and we shall without further comment use relations like this.

To obtain non-trivial limits as $u \to \infty$ we introduce the rescaled process

$$\xi_u(t) = u(\xi(tq) - u),$$

where we shall let q tend to zero as $u \to \infty$. Here we have to be a little more specific about this convergence than in Chapter 7, and shall assume that $u^{2/\alpha}q \to a > 0$, and let a tend to zero at a later stage.

<u>LEMMA 12.2</u> Suppose $u \to \infty$, $q \to 0$ so that $u^{2/\alpha}q \to a > 0$. Then

(i) the conditional distributions of $\xi_u(t)$ given that $\xi_u(0) = x$, are normal with

$$E(\xi_u(t) \mid \xi_u(0)=x) = x - Ca^{\alpha}|t|^{\alpha}(1+o(1)),$$

$$\mathrm{Cov}(\xi_u(s), \xi_u(t) \mid \xi_u(0)=x) = Ca^{\alpha}(|s|^{\alpha} + |t|^{\alpha} - |t-s|^{\alpha}) + o(1)$$

where for fixed x the $o(1)$ are uniform for $\max(|s|,|t|) \le t_0$, for all $t_0 > 0$,

(ii) for all $t_0 > 0$ there is a constant K, not depending on a or x, such that, for $|s|,|t| \le t_0$,

$$\mathrm{Var}(\xi_u(s) - \xi_u(t) \mid \xi_u(0)=x) \le Ka^{\alpha}|t-s|^{\alpha}.$$

<u>PROOF</u> (i) Since the process $\{\xi_u(t)\}$ is normal with mean $-u^2$ and covariance function

$$\mathrm{Cov}(\xi_u(s), \xi_u(t)) = u^2 r((t-s)q)$$

we obtain (see e.g. Rao (1973), p. 522) that the conditional distributions are normal with

$$E(\xi_u(t)\,|\,\xi_u(0)=x) = -u^2 + u^2 r(tq)\cdot\frac{1}{u^2}(x+u^2)$$

$$= -u^2 + (1 - Cq^\alpha|t|^\alpha + o(q^\alpha|t|^\alpha))(x+u^2)$$

$$= x - (x+u^2)(Cq^\alpha|t|^\alpha + o(q^\alpha|t|^\alpha))$$

$$= x - Ca^\alpha|t|^\alpha(1+o(1)),$$

since $u^2q^\alpha \to a^\alpha > 0$ and x is fixed. Furthermore,

$$\mathrm{Cov}(\xi_u(s),\ \xi_u(t)\,|\,\xi_u(0)=x) = u^2(r((t-s)q) - r(sq)r(tq))$$

$$= u^2(1 - Cq^\alpha|t-s|^\alpha - (1-Cq^\alpha|s|^\alpha)(1-Cq^\alpha|t|^\alpha) + o(q^\alpha))$$

$$= Ca^\alpha(|s|^\alpha + |t|^\alpha - |t-s|^\alpha) + o(1),$$

uniformly for $\max(|s|,|t|) \le t_0$.

(ii) Since $\xi_u(s) - \xi_u(t)$ and $\xi_u(0)$ are normal with variances $2u^2(1-r((t-s)q))$ and u^2, respectively, and covariance $u^2(r(tq) - r(sq))$, we have, for some constant K,

$$\mathrm{Var}(\xi_u(s) - \xi_u(t)\,|\,\xi_u(0)=x)$$

$$= 2u^2(1-r((t-s)q)) - u^2(r(tq)-r(sq))^2$$

$$\le 2Cu^2q^\alpha|t-s|^\alpha + o(u^2q^\alpha|t-s|^\alpha)$$

$$\le Ka^\alpha|t-s|^\alpha,$$

for $|s|,|t| \le t_0$. □

The first step in the derivation of the tail distribution of $M(h) = \sup\{\xi(t);\ 0\le t\le h\}$ is to consider the maximum taken over a fixed number of points, $0, q, \ldots, (n-1)q$.

<u>LEMMA 12.3</u> For each C there is a constant $h_\alpha(n,a) < \infty$ such that, if $u \to \infty$, $q \to 0$, $u^{2/\alpha}q \to a > 0$, then

$$\frac{1}{\phi(u)/u} P\{\max_{0\le j<n} \xi(jq) > u\} \to C^{1/\alpha} H_\alpha(n,a).$$

<u>PROOF</u> We have

$$P\{\max_{0\le j<n} \xi(jq) > u\} = P\{\max_{0\le j<n} \xi_u(j) > 0\}$$

$$= P\{\xi_u(0) > 0\} + P\{\xi_u(0) \le 0, \max_{0<j<n} \xi_u(j) > 0\},$$

where $P\{\xi_u(0) > 0\} = P\{\xi(0) > u\} = 1 - \Phi(u) \sim \phi(u)/u$. Since furthermore $\xi_u(0)$ is normal with mean $-u^2$ and variance u^2, we have

$$P\{\xi_u(0) \le 0, \max_{0<j<n} \xi_u(j) > 0\}$$

$$(12.7) \qquad = \int_{-\infty}^{0} \frac{1}{u} \phi(u + \frac{x}{u}) P\{\max_{0<j<n} \xi_u(j) > 0 | \xi_u(0) = x\} dx$$

$$= \frac{\phi(u)}{u} \int_{-\infty}^{0} e^{-x-x^2/2u^2} P\{\max_{0<j<n} (\xi_u(j) - x) > -x | \xi_u(0) = x\} dx.$$

By Lemma 12.2 (i), for any fixed x,

$$E(\xi_u(j) - x | \xi_u(0) = x) \to -Ca^\alpha |j|^\alpha,$$

$$\mathrm{Cov}(\xi_u(i) - x, \xi_u(j) - x | \xi_u(0) = x) \to Ca^\alpha (|i|^\alpha + |j|^\alpha - |i-j|^\alpha)$$

as $q \to 0$. Since limits of covariances are covariances, one can define a sequence of normal r.v.'s, $Y_a(j)$, $j = 1, 2, \ldots$ with mean and covariances depending on $a = \lim qu^{2/\alpha}$,

$$E(Y_a(j)) = -Ca^\alpha |j|^\alpha,$$

$$\mathrm{Cov}(Y_a(i), Y_a(j)) = Ca^\alpha (|i|^\alpha + |j|^\alpha - |i-j|^\alpha).$$

Now convergence of moments implies convergence in distribution for jointly normal r.v.'s (as can easily be seen, e.g. using characteristic functions). The boundary of the set $\{\max_{0<j<n} Y_a(j) > -x\}$ is contained in the set

$$\bigcup_{j=1}^{n-1} \{Y_a(j) = -x\}$$

and since the one-dimensional distributions of $Y_a(1), \ldots, Y_a(n-1)$ are all non-degenerate, $\{\max_{0<j<n} Y_a(j) > -x\}$ is a.s. a continuity set, and it follows that

$$P\{\max_{0<j<n} (\xi_u(j)-x) > -x \mid \xi_u(0)=x\} \to P\{\max_{0<j<n} Y_a(j) > -x\}.$$

To be able to use the dominated convergence theorem in (12.7) we note that, by Lemma 12.2 (i),

$$P\{\max_{0<j<n} (\xi_u(j)-x) > -x \mid \xi_u(0)=x\}$$

$$\leq \sum_{j=1}^{n} P\{\xi_u(j)-x > -x \mid \xi_u(0)=x\}$$

$$\leq n(1-\Phi(c'-c''x)) \leq n\,\frac{\phi(c'-c''x)}{c'-c''x}$$

for come constants $c', c'' > 0$. This shows that the convergence in (12.7) is dominated, and we obtain

$$\frac{1}{\phi(u)/u} P\{\max_{0\leq j<n} \xi(jq) > u\} \to 1 + \int_{-\infty}^{0} e^{-x} P\{\max_{0<j<n} Y_a(j) > -x\}dx < \infty,$$

which proves the existence and finiteness of the constant $H_\alpha(n,a)$. □

For future use we note the following expression for the constant $H_\alpha(n,a)$:

$$H_\alpha(n,a) = C^{-1/\alpha}\left(1 + \int_{-\infty}^{0} e^{-x} P\{\max_{0<j<n} Y_a(j) > -x\}dx\right). \tag{12.8}$$

<u>LEMMA 12.4</u> Suppose $u \to \infty$, $q \to 0$, $u^{2/\alpha}q \to a > 0$, and take h such that $\sup_{\varepsilon\leq t\leq h} r(t) < 1$ for all $\varepsilon > 0$. Then, for each C,

(i) there is a constant $H_\alpha(a) < \infty$ such that

$$\frac{H_\alpha(n,a)}{na} \to H_\alpha(a) \quad \text{as} \quad n \to \infty,$$

and

$$\frac{1}{u^{2/\alpha}\phi(u)/u} P\{\max_{0\leq jq\leq h} \xi(jq) > u\} \to h\, C^{1/\alpha} H_\alpha(a),$$

(ii) $H_\alpha(a_0) > 0$ for some $a_0 > 0$.

PROOF (i) Let n be a fixed integer, write $m = [h/nq]$, and

$$B_r = \{\max_{rn \le j < (r+1)n} \xi(jq) > u\}.$$

Then

$$P(\bigcup_{r=0}^{m-1} B_r) \le P\{\max_{0 \le jq \le h} \xi(jq) > u\} \le P(\bigcup_{r=0}^{m} B_r),$$

where, by Lemma 12.3,

$$P(\bigcup_{r=0}^{m} B_r) \le (m+1)P(B_0) \sim (m+1)\frac{\phi(u)}{u} C^{1/\alpha} H_\alpha(n,a)$$

$$\sim \frac{hu^{2/\alpha}}{na}\frac{\phi(u)}{u} C^{1/\alpha} H_\alpha(n,a)$$

since $1/q \sim u^{2/\alpha}/a$ by assumption. Hence

(12.9) $$\limsup_{u\to\infty} \frac{1}{u^{2/\alpha}\phi(u)/u} P\{\max_{0 \le jq \le h} \xi(jq) > u\} \le h\, C^{1/\alpha} \frac{H_\alpha(n,a)}{na} < \infty.$$

Furthermore

$$P(\bigcup_{r=0}^{m-1} B_r) \ge \sum_{r=0}^{m-1} P(B_r) - \sum_{r \ne s} P(B_r \cap B_s)$$

$$\ge nP(B_0) - m\sum_{r=1}^{m-1} P(B_0 \cap B_r)$$

so that

(12.10) $$\liminf_{u\to\infty} \frac{1}{u^{2/\alpha}\phi(u)/u} P(\bigcup_{r=0}^{m-1} B_r) \ge h\, C^{1/\alpha} \frac{H(n,a)}{na} -$$

$$- \limsup_{u\to\infty} \frac{1}{u^{2/\alpha}\phi(u)/u} m \sum_{r=1}^{m-1} P(B_0 \cap B_r)$$

$$= h\, C^{1/\alpha} \frac{H(n,a)}{na}, \text{ say.}$$

We shall now show that

(12.11) $$\rho_n = \limsup_{u\to\infty} \frac{1}{u^{2/\alpha}\phi(u)/u} m \sum_{r=1}^{m-1} P(B_0 \cap B_r) \to 0 \quad \text{as} \quad n \to \infty.$$

By Boole's inequality and stationarity

$$m \sum_{r=1}^{m-1} P(B_0 \cap B_r) =$$

(12.12)
$$\begin{aligned}
&= m \sum_{r=1}^{m-1} P\Big(\bigcup_{0\le i<n} \ \bigcup_{rn\le j<(r+1)n} \{\xi(iq) > u,\ \xi(jq) > u\}\Big) \\
&\le m \sum_{i=0}^{n-1} \sum_{j=n}^{mn} P\{\xi(iq) > u,\ \xi(jq) > u\} \\
&\le m \sum_{j=1}^{n} j\, P\{\xi(0) > u,\ \xi(jq) > u\} \\
&\qquad + mn \sum_{j=n+1}^{mn} P\{\xi(0) > u,\ \xi(jq) > u\}.
\end{aligned}$$

To estimate these sums we have to use different techniques for small and large values of jq. Let $\varepsilon > 0$ be such that

$$1 \ge 1 - r(t) \ge \frac{C}{2} |t|^{\alpha} \quad \text{for} \quad |t| \le \varepsilon.$$

Assume $jq \le \varepsilon$, and write $r = r(jq)$. Since the conditional distribution of $\xi(jq)$ given $\xi(0) = x$, is normal with mean rx and variance $1 - r^2$,

$$P\{\xi(0) > u,\ \xi(jq) > u\} = \int_u^{\infty} \phi(x) P\{\xi(jq) > u \mid \xi(0) = x\} dx$$

$$= \int_u^{\infty} \phi(x) \Big(1 - \Phi\Big(\frac{u-xr}{\sqrt{1-r^2}}\Big)\Big) dx$$

(12.13)
$$\begin{aligned}
&= \Big[-(1-\Phi(x))\Big(1-\Phi\Big(\frac{u-xr}{\sqrt{1-r^2}}\Big)\Big)\Big]_{x=u}^{\infty} + \int_u^{\infty} (1-\Phi(x)) \frac{r}{\sqrt{1-r^2}} \phi\Big(\frac{u-xr}{\sqrt{1-r^2}}\Big) dx \\
&\le (1-\Phi(u))\Big(1-\Phi\Big(u\sqrt{\tfrac{1-r}{1+r}}\Big)\Big) + \int_u^{\infty} (1-\Phi(u)) \frac{r}{\sqrt{1-r^2}} \phi\Big(\frac{u-xr}{\sqrt{1-r^2}}\Big) dx \\
&= 2(1-\Phi(u))\Big(1-\Phi\Big(u\sqrt{\tfrac{1-r}{1+r}}\Big)\Big) \le 2\,\frac{\phi(u)}{u}\Big(1-\Phi\Big(u\sqrt{\tfrac{1-r}{1+r}}\Big)\Big).
\end{aligned}$$

Here,

(12.14) $$\frac{1-r}{1+r} = \frac{1-r(jq)}{1+r(jq)} \ge \frac{C}{4} |jq|^{\alpha} \ge K j^{\alpha} a^{\alpha} u^{-2},$$

for some constant $K > 0$, and thus if $nq \le \varepsilon$,

$$m \sum_{j=1}^{n} j\, P\{\xi(0) > u,\ \xi(jq) > u\}$$

$$\leq 2m \sum_{j=1}^{n} j\, \frac{\phi(u)}{u}(1 - \Phi(\sqrt{Kj^\alpha a^\alpha}))$$

(12.15)

$$\sim \frac{2h}{q}\, \frac{\phi(u)}{u} \cdot \frac{1}{n} \sum_{j=1}^{n} j(1 - \Phi\sqrt{Kj^\alpha a^\alpha}))$$

$$\leq \frac{u^{2/\alpha}\phi(u)}{u} \cdot \frac{K'h}{a}\, \frac{1}{n} \sum_{j=1}^{\infty} j(1 - \Phi\sqrt{Kj^\alpha a^\alpha}))$$

for some constant K'. Since $1 - \Phi(x) \leq \phi(x)/x$ the sum

$$\sum_{j=1}^{\infty} j(1 - \Phi(\sqrt{Kj^\alpha a^\alpha}))$$

is convergent, and since $nq \to 0$ as $u \to \infty$ (n fixed)

$$\limsup_{u\to\infty} \frac{1}{u^{2/\alpha}\phi(u)/u}\, m \sum_{j=1}^{n} j\, P\{\xi(0) > u,\ \xi(jq) > u\} = O(1/n) \tag{12.16}$$

as $n \to \infty$.

For the second sum in (12.12) we get, again using (12.13),

$$mn \sum_{j=n+1}^{[\varepsilon/q]} P\{\xi(0) > u,\ \xi(jq) > u\}$$

$$\leq \frac{K''h}{qu^{2/\alpha}}\, \frac{u^{2/\alpha}\phi(u)}{u} \sum_{j=n+1}^{\infty} (1 - \Phi(\sqrt{Kj^\alpha a^\alpha})). \tag{12.17}$$

where the sum is convergent. For terms with $jq \geq \varepsilon$ we use the estimate from Lemma 3.2,

$$P\{\xi(0) > u,\ \xi(jq) > u\} \leq (1 - \Phi(u))^2 + Ke^{-u^2/(1+|r(jq)|)},$$

which implies that

$$mn \sum_{j=[\varepsilon/q]+1}^{mn} P\{\xi(0) > u,\ \xi(jq) > u\}$$

$$\leq \left(\frac{h}{q}\right)^2 (1 - \Phi(u))^2 + \frac{Kh}{q} \sum_{\varepsilon < jq \leq h} e^{-u^2/(1+|r(jq)|)}.$$

Since $\delta = \sup_{\varepsilon \leq t \leq h} |r(t)| < 1$, this is bounded by

$$\frac{h^2}{q^2}(1-\Phi(u))^2+\frac{Kh^2}{q^2}e^{-u^2/(1+\delta)}$$

$$=\frac{u^{2/\alpha}\phi(u)}{u}\left(K\frac{u}{u^{2/\alpha}q^2}e^{-\frac{u^2}{2}\cdot\frac{1-\delta}{1+\delta}}+o(1)\right)$$

$$(12.18)\qquad =\frac{u^{2/\alpha}\phi(u)}{u}\cdot o(1)\quad \text{as}\quad u\to\infty,$$

since $u/(u^{2/\alpha}q^2)\sim u^{1+2/\alpha}/a^2$ and $\delta<1$.

Together, (12.17) and (12.18) imply that

$$\limsup_{u\to\infty}\frac{1}{u^{2/\alpha}\phi(u)/u}\, mn\sum_{j=n+1}^{mn} P\{\xi(0)>u,\ \xi(jq)>u\}$$

$$=K'\sum_{j=n+1}^{\infty}(1-\Phi(\sqrt{Kj^{\alpha}a^{\alpha}}))\to 0,\ n\to\infty,$$

and combining this with (12.16) and (12.12) we obtain (12.11).

Thus we have shown that

$$h\,C^{1/\alpha}\cdot\frac{H_\alpha(n,a)}{na}-\rho_n\le\liminf_{u\to\infty}\frac{1}{u^{2/\alpha}\phi(u)/u}P\{\max_{0\le jq\le h}\xi(jq)>u\}$$

$$\le\limsup_{u\to\infty}\frac{1}{u^{2/\alpha}\phi(u)/u}P\{\max_{0\le jq\le h}\xi(jq)>u\}$$

$$\le h\,C^{1/\alpha}\cdot\frac{H_\alpha(n,a)}{na}$$

where $\rho_n\to 0$ as $n\to\infty$. Since $H_\alpha(n,a)<\infty$ for all n, and the lim inf and lim sup do not depend on n, this implies the existence of

$$\lim_{n\to\infty}\frac{H_\alpha(n,a)}{na}=H_\alpha(a),$$

which is then the joint value of lim inf and lim sup. Furthermore this proves that $H_\alpha(a)<\infty$.

(ii) Take $\varepsilon>0$ small enough to make

$$1\ge 1-r(t)\ge 2K|t|^{\alpha},\ \text{some}\ K>0,$$

for $|t|\le\varepsilon$. Applying (12.15), we obtain for $|h|\le\varepsilon$,

$$P\{\max_{0\le jq\le h} \xi(jq) > u\}$$

$$\ge [\frac{h}{q}]\, P\{\xi(0) > u\} - [\frac{h}{q}] \sum_{0<jq\le h} P\{\xi(0) > u,\ \xi(jq) > u\}$$

$$\ge [\frac{h}{q}](1-\Phi(u)) - 2[\frac{h}{q}] \sum_{0<jq\le h} \frac{\phi(u)}{u} \left(1-\Phi\left(u\sqrt{\frac{1-r(jq)}{1+r(jq)}}\right)\right)$$

$$\ge [\frac{h}{q}]\, \frac{\phi(u)}{u}(1+o(1) - 2\sum_{j=1}^{[h/q]+1} (1-\Phi\sqrt{Kj^\alpha a^\alpha})).$$

Since $[\frac{h}{q}] \sim \frac{h}{a} u^{2/\alpha}$, part (ii) follows from

$$\sum_{j=1}^{[h/q]+1} (1-\Phi(\sqrt{Kj^\alpha a^\alpha})) \le \sum_{j=1}^{\infty} (1-\Phi(\sqrt{Kj^\alpha a^\alpha}))$$

$$\le \sum_{j=1}^{\infty} \frac{\phi(\sqrt{Kj^\alpha a^\alpha})}{\sqrt{Kj^\alpha a^\alpha}} \to 0 \quad \text{as} \quad a \to \infty,$$

as there is then certainly one a_0 that makes the sum less than 1/2. □

The following three lemmas relate the continuous maximum $\sup_{0\le t\le h} \xi(t)$ to the discrete one $\max_{0\le jq\le h} \xi(jq)$. We first prove that we can neglect the probability that the discrete maximum is less than $u - \gamma/u$ and the continuous is greater than u, as $\gamma = a^\beta \to 0$.

<u>LEMMA 12.5</u> Let $u \to \infty$, $q \to 0$, $u^{2/\alpha}q \to a > 0$, and let $\gamma = a^\beta$ for some positive constant $\beta < \alpha/2$. Then

$$\nu_a = \limsup_{u\to\infty} \frac{1}{u^{2/\alpha}\phi(u)/u} P\{\max_{0\le jq\le h} \xi(jq) \le u - \frac{\gamma}{u},\ \sup_{0\le t\le h} \xi(t) > u\}$$

$$\to 0 \quad \text{as} \quad a \to 0.$$

<u>PROOF</u> By Boole's inequality and stationarity

$$P\{\max_{0\le jq\le h} \xi(jq) \le u - \frac{\gamma}{u},\ \sup_{0\le t\le h} \xi(t) > u\}$$

$$\le \frac{h}{q} P\{\xi(0) \le u - \frac{\gamma}{u},\ \sup_{0\le t\le q} \xi(t) > u\},$$

and with $\xi_u(s) = u(\xi(sq) - u)$, we can write

$$P\{\xi(0) \le u - \tfrac{\gamma}{u}, \sup_{0\le t\le q} \xi(t) > u\}$$

$$= \int_{x=-\infty}^{u-\gamma/u} \phi(x) P\{\sup_{0\le t\le q} \xi(t) > u \mid \xi(0) = x\} dx$$

$$= \int_{y=-\infty}^{-\gamma} \frac{1}{u} \phi(u + \tfrac{y}{u})\, P\{\sup_{0\le s\le 1} \xi_u(s) > 0 \mid \xi_u(0) = y\} dy.$$

By Lemma 12.2 (i), the conditional distributions of $\xi_u(s)$ given $\xi_u(0) = y$ are normal with mean

$$\mu(s) = y - a^{\alpha}|s|^{\alpha}(1 + o(1)) \quad \text{as} \quad q \to 0$$

with the $o(1)$ uniform in $|s| \le 1$. Here $\mu(s) < y$ for small q, and

$$P\{\sup_{0\le s\le 1} \xi_u(s) > 0 \mid \xi_u(0) = y\} \le P\{\sup_{0\le s\le 1} (\xi_u(s) - \mu(s)) > -y \mid \xi_u(0) = y\},$$

where, conditional on $\xi_u(0)$, $\xi_u(s) - \mu(s)$ is a non-stationary normal process with mean zero and, by Lemma 12.2 (ii), incremental variance

$$\mathrm{Var}(\xi_u(s) - \xi_u(t) \mid \xi_u(0) = y) \le Ka^{\alpha}|t - s|^{\alpha},$$

for some constant K which is independent of a and y. Fernique's Lemma (Lemma 12.1) implies that, with $c = c_{\alpha}/K$,

$$P\{\sup_{0\le s\le 1} (\xi_u(s) - \mu(s)) > -y \mid \xi_u(0) = y\} \le 4\exp(-ca^{-\alpha}y^2),$$

and thus, using K and c as generic constants,

$$\frac{1}{u^{2/\alpha}\phi(u)/u} P\{\max_{0\le jq\le h} \xi(jq) \le u - \tfrac{\gamma}{u}, \sup_{0\le t\le h} \xi(t) > u\}$$

$$\le \frac{h}{qu^{2/\alpha}\phi(u)} \int_{-\infty}^{-\gamma} \phi(u + \tfrac{y}{u}) \cdot \exp(-ca^{-\alpha}y^2) dy$$

$$\le \frac{K}{qu^{2/\alpha}} \int_{-\infty}^{-\gamma} e^{-y-ca^{-\alpha}y^2} dy$$

$$\le \frac{K}{qu^{2/\alpha}} \int_{-\infty}^{-\gamma} e^{-ca^{-\alpha}y^2} dy$$

$$\sim Ka^{\alpha/2-1}\, \phi(-ca^{\beta-\alpha/2}).$$

Clearly, this tends to zero as $a \to 0$, since $\beta < \alpha/2$, which proves the lemma. $\square$

LEMMA 12.6 If $u \to \infty$, $q \to 0$, $u^{2/\alpha}q \to a > 0$, and $\gamma = a^{\beta}$ for some constant $\beta > 0$, then, with h as in Lemma 12.4,

$$\lim_{u\to\infty} \frac{1}{u^{2/\alpha}\phi(u)/u} P\{u - \frac{\gamma}{u} < \max_{0\le jq\le h} \xi(jq) \le u\} = h(e^{\gamma} - 1)C^{1/\alpha} H_{\alpha}(a).$$

PROOF Since $u^{2/\alpha}q \to a > 0$ implies $(u - \frac{\gamma}{u})^{2/\alpha}q \to a$, and since furthermore

$$(u - \frac{\gamma}{u})^{2/\alpha} \frac{\phi(u - \frac{\gamma}{u})}{u - \frac{\gamma}{u}} \sim e^{\gamma} \frac{u^{2/\alpha}\phi(u)}{u}$$

as $u \to \infty$, it follows from Lemma 12.4 (i) that

$$\frac{1}{u^{2/\alpha}\phi(u)/u} P\{u - \frac{\gamma}{u} < \max_{0\le jq\le h} \xi(jq) \le u\}$$

$$= \frac{1}{u^{2/\alpha}\phi(u)/u} (P\{\max_{0\le jq\le h} \xi(jq) > u - \frac{\gamma}{u}\} - P\{\max_{0\le jq\le h} \xi(jq) > u\})$$

$$\to he^{\gamma}C^{1/\alpha}H_{\alpha}(a) - hC^{1/\alpha}H_{\alpha}(a).$$ □

LEMMA 12.7 Under the conditions of Lemma 12.4,

(i) $$h\, C^{1/\alpha} H_{\alpha}(a) \le \liminf_{u\to\infty} \frac{1}{u^{2/\alpha}\phi(u)/u} P\{\sup_{0\le t\le h} \xi(t) > u\}$$

(12.19) $$\le \limsup_{u\to\infty} \frac{1}{u^{2/\alpha}\phi(u)/u} P\{\sup_{0\le t\le h} \xi(t) > u\}$$

$$\le \nu_a + h(e^{\gamma} - 1)C^{1/\alpha}H_{\alpha}(a) + hC^{1/\alpha}H_{\alpha}(a) < \infty,$$

for $\gamma = a^{\beta}$, where, by Lemma 12.5, $\nu_a \to 0$ as $a \to 0$.

(ii) $\lim_{a\to 0} H_{\alpha}(a) = H_{\alpha}$, say

exists finite, and

(12.20) $$\frac{1}{u^{2/\alpha}\phi(u)/u} P\{\sup_{0\le t\le h} \xi(t) > u\} \to hC^{1/\alpha}H_{\alpha}.$$

(iii) H_{α} is independent of C.

PROOF Since

$$P\{\max_{0\le jq\le h} \xi(jq) > u\} \le P\{\sup_{0\le t\le h} \xi(t) > u\}$$

$$\leq P\{\max_{0\leq jq\leq h} \xi(jq) \leq u-\frac{\gamma}{u}, \sup_{0\leq t\leq h} \xi(t) > u\}$$

$$+ P\{u-\frac{\gamma}{u} < \max_{0\leq jq\leq h} \xi(jq) \leq u\}$$

$$+ P\{\max_{0\leq jq\leq h} \xi(jq) > u\},$$

part (i) follows directly from Lemmas 12.4, 12.5 and 12.6.

Further, the middle limits in (12.19) are independent of a, and it follows that $\limsup_{a\to 0} H_\alpha(a) < \infty$. Therefore

$$h(e^\gamma - 1)C^{1/\alpha}H_\alpha(a) \to 0$$

as $a \to 0$, and since $\nu_a \to 0$ it follows as in the proof of Lemma 12.4 (i) that $\lim_{a\to 0} H_\alpha(a)$ exists, finite and (12.20) holds.

For part (iii), note that if $\xi(t)$ satisfies (2.1) then the covariance function $\tilde{r}(\tau)$ of $\tilde{\xi}(t) = \xi(t/C^{1/\alpha})$ satisfies

$$\tilde{r}(\tau) = 1 - |\tau|^\alpha + o(|\tau|^\alpha) \quad \text{as} \quad \tau \to 0.$$

Furthermore

$$\frac{1}{u^{2/\alpha}\phi(u)/u} P\{\sup_{0\leq t\leq h} \xi(t) > u\} = \frac{1}{u^{2/\alpha}\phi(u)/u} P\{\sup_{0\leq t\leq hC^{1/\alpha}} \tilde{\xi}(t) > u\},$$

which by (ii) shows that H_α does not depend on C. □

One immediate consequence of (12.20) and Lemma 12.4 (i) is that

$$\limsup_{u\to\infty} \frac{1}{u^{2/\alpha}\phi(u)/u} P\{\max_{0\leq jq\leq h} \xi(jq) \leq u, \sup_{0\leq t\leq h} \xi(t) > u\}$$

$$(12.21) \qquad = \limsup_{u\to\infty} \left(\frac{1}{u^{2/\alpha}\phi(u)/u} P\{\sup_{0\leq t\leq h} \xi(t) > u\} - \frac{1}{u^{2/\alpha}\phi(u)/u} P\{\max_{0\leq jq\leq h} \xi(jq) > u\}\right)$$

$$= hC^{1/\alpha}(H_\alpha - H_\alpha(a)) \to 0 \quad \text{as} \quad a \to 0.$$

Of course, (12.20) has its main interest if $H_\alpha > 0$, but to prove this requires some further work.

<u>LEMMA 12.8</u> $H_\alpha > 0$.

<u>PROOF</u> We have from Lemma 12.4 (i) and (ii) that there is an $a_0 > 0$ such that

$$H_\alpha(a_0) = \lim_{n\to\infty} \frac{H_\alpha(n,a_0)}{na_0} > 0.$$

Let the sequence $Y_a(j)$ be as in the proof of Lemma 12.3, i.e. normal with mean $-Ca^\alpha|j|^\alpha$ and covariances $Ca^\alpha(|i|^\alpha + |j|^\alpha - |j-i|^\alpha)$. Then we have from (12.7),

$$C^{1/\alpha}H_\alpha(n,a) = 1 + \int_{-\infty}^{0} e^{-x} P\{\max_{0<j<n} Y_a(j) > -x\}dx,$$

$$C^{1/\alpha}H_\alpha(nk,a) = 1 + \int_{-\infty}^{0} e^{-x} P\{\max_{0<j<nk} Y_a(j) > -x\}dx,$$

$$C^{1/\alpha}H_\alpha(n,ak) = 1 + \int_{-\infty}^{0} e^{-x} P\{\max_{0<j<n} Y_{ak}(j) > -x\}dx.$$

Here $Y_a(jk)$, $j=1, \ldots, n$ have the same distributions as $Y_{ak}(j)$, $j=1, \ldots, n$, which implies

$$H_\alpha(n,ak) \le H_\alpha(nk,a)$$

for $k=1, 2, \ldots$, the r.v.'s in $H_\alpha(n,ak)$ forming a subset of those appearing in $H_\alpha(nk,a)$. Thus

$$0 < H_\alpha(a_0) = \lim_{n\to\infty} \frac{H_\alpha(n,a_0)}{na_0} \le \lim_{n\to\infty} \frac{H_\alpha(nk,a_0/k)}{nk\cdot(a_0/k)} = H_\alpha(a_0/k),$$

and since $H_\alpha(a_0/k) \to H_\alpha$ as $k \to \infty$, the lemma follows. □

By combining Lemmas 12.7 and 12.8 we obtain the tail of the distribution of the maximum $M(h) = \sup\{\xi(t);\ 0 \le t \le h\}$ over a fixed interval.

<u>THEOREM 12.9</u> If $r(t)$ satisfies (12.1), then for each fixed $h > 0$ such that $\sup_{\varepsilon\le t\le h} r(t) = \delta_\varepsilon < 1$ for all $\varepsilon > 0$,

$$\lim_{u\to\infty} \frac{1}{u^{2/\alpha}\phi(u)/u} P\{M(h) > u\} = h\, C^{1/\alpha} H_\alpha,$$

where $H_\alpha > 0$ is a finite constant depending only on α. □

REMARK 12.10 In the proof of Theorem 12.9 we obtained the *existence* of the constant H_α by rather tricky estimates, starting with

$$H_\alpha(n,a) = C^{-1/\alpha}\left(1 + \int_{-\infty}^{0} e^{-x} P\{\max_{0\le j\le n} Y_a(j) > -x\} dx\right).$$

By pursuing these estimates further one can obtain a related expression for H_α,

$$H_\alpha = \lim_{T\to\infty} T^{-1} \int_{-\infty}^{0} e^{-x} P\{\sup_{0\le t\le T} Y_0(t) > -x\} dx,$$

where $\{Y_0(t)\}$ is a non-stationary normal process with mean $-|t|^\alpha$ and covariances $|s|^\alpha + |t|^\alpha - |t-s|^\alpha$. However, this does not seem to be very instructive, nor of much help in computing H_α.

It should be noted, though, that the proper time-normalization of the maximum distribution only depends on the covariance function through the time-scale $C^{1/\alpha}$ and on the constant H_α. Therefore, if one can find the limiting form of the tail of the distribution of $M(h)$ (for some h) for one single process satisfying (12.1) one also knows the value of H_α for that particular α. For $\alpha = 2$ this is easily done, by considering the simple cosine-process (6.6). By comparing (6.12) and Theorem 12.9, we find $H_2 = 1/\sqrt{\pi}$.

The only other value of α for which the tail of the distribution of $M(h)$ has been found is $\alpha = 1$. In fact, explicit expressions for the entire distribution of $M(h)$ are known for the normal process with triangular covariance function $r(t) = 1 - |t|$, $|t| \le 1$, see Slepian (1961), and as a result one has $H_1 = 1$. In particular, this shows that for the Ornstein-Uhlenbeck process, with $r(t) = \exp(-|t|)$, $P\{M(h) > u\} \sim hu\phi(u)$. □

Before proceeding to the maxima over increasing intervals we formulate the following lemma for later reference.

LEMMA 12.11 Suppose $\{\xi(t)\}$ satisfies (12.1), let $h > 0$ be fixed such that $\sup_{\varepsilon\le t\le h} r(t) < 1$ for all $\varepsilon > 0$, and let $u \to \infty$, $q \to 0$,

$u^{2/\alpha}q \to a > 0$. Then for every interval I of length h,

$$0 \le P\{\xi(jq) \le u,\ jq \in I\} - P\{M(I) \le u\} \le \mu h \rho_a + o(\mu),$$

where $\mu = C^{1/\alpha}H_\alpha u^{2/\alpha}\phi(u)/u$, $\rho_a = 1 - H_\alpha(a)/H_\alpha \to 0$ as $a \to 0$, and the $o(\mu)$-term is the same for all intervals of length h.

<u>PROOF</u> By stationarity

$$0 \le P\{\xi(jq) \le u,\ jq \in I\} - P\{M(I) \le u\}$$
$$\le P\{\xi(0) > u\} + P\{\xi(jq) \le u,\ jq \in [0,h]\} - P\{M(h) \le u\},$$

where $P\{\xi(0) > u\} \le \phi(u)/u = o(\mu)$. Therefore the result is immediate from Lemma 12.4 (i),

$$\mu^{-1} P\{\max_{0 \le jq \le h} \xi(jq) > u\} = hH_\alpha(a)/H_\alpha + o(1),$$

and (12.20),

$$\mu^{-1} P\{M(h) > u\} = h + o(1). \qquad \square$$

<u>Maxima over increasing intervals</u>

The covariance condition (7.2), i.e. $r(t)\log t \to 0$ as $t \to \infty$, is also sufficient to establish the double exponential limit for the maximum $M(T) = \sup\{\xi(t);\ 0 \le t \le T\}$ in this general case. We then let $T \to \infty$, $u \to \infty$ so that

$$T\mu = TC^{1/\alpha}H_\alpha u^{2/\alpha}\phi(u)/u \to \tau > 0,$$

i.e. $TP\{M(h) > u\} \to \tau h$. Taking logarithms we get

$$\log T + \log(C^{1/\alpha}H_\alpha(2\pi)^{-1/2}) + \frac{2-\alpha}{\alpha}\log u - \frac{u^2}{2} \to \log\tau,$$

implying

$$u^2 \sim 2\log T,$$

or $\log u = \frac{1}{2}\log 2 + \frac{1}{2}\log\log T + o(1)$, which gives

(12.22) $u^2 = 2\log T + \frac{2-\alpha}{\alpha}\log\log T - 2\log\tau$

$$+ 2\log\left(C^{1/\alpha}H_\alpha(2\pi)^{-1/2}2^{(2-\alpha)/2\alpha}\right) + o(1).$$

<u>LEMMA 12.12</u> Let $\varepsilon > 0$ be given, and suppose (12.1) and 7.2) both hold. Let $T \sim \tau/\mu$ for $\tau > 0$ fixed and with $\mu = C^{1/\alpha}H_\alpha u^{2/\alpha}\phi(u)/u$, so that $u \sim (2\log T)^{1/2}$ as $T \to \infty$, and let $q \to 0$ as $u \to \infty$ in such a way that $u^{2/\alpha}q \to a > 0$. Then

(12.23) $\frac{T}{q}\sum_{\varepsilon \le kq \le T} |r(kq)|e^{-u^2/(1+|r(kq)|)} \to 0$ as $T \to \infty$.

<u>PROOF</u> This lemma corresponds to Lemma 7.1. First, we split the sum in (12.23) at T^β, where β is a constant such that $0 < \beta < \frac{1-\delta}{1+\delta}$, $\delta = \sup\{|r(t)|;\ |t| \ge \varepsilon\} < 1$. Then, with the constant K changing from line to line,

$$\frac{T}{q}\sum_{\varepsilon \le kq \le T^\beta} |r(kq)|e^{-u^2/(1+|r(kq)|)} \le \frac{T^{\beta+1}}{q^2}e^{-u^2/(1+\delta)}$$

$$\le \frac{K}{q^2}T^{\beta+1-\frac{2}{1+\delta}} \sim \frac{K}{(u^{2/\alpha}q)^2}(\log T)^{2/\alpha}T^{\beta+1-\frac{2}{1+\delta}}$$

$$\to 0 \text{ as } T \to \infty,$$

since $\exp(-u^2/2) \le K/T$, $u^2 \sim 2\log T$, and $u^{2/\alpha}q \to a > 0$.

With $\delta(t) = \sup\{|r(s)\log s|;\ s \ge t\}$, we have $|r(t)| \le \frac{\delta(t)}{\log t}$ as $t \to \infty$, and hence for $kq \ge T^\beta$,

$$e^{-u^2/(1+|r(kq)|)} \le e^{-u^2(1-\delta(T^\beta)/\log T^\beta)},$$

so that the remaining sum is bounded by

(12.24) $\frac{T}{q}\sum_{T^\beta \le kq \le T} |r(kq)|e^{-u^2(1-\delta(T^\beta)/\log T^\beta)}$

$$\le \left(\frac{T}{q}\right)^2 e^{-u^2(1-\delta(T^\beta)/\log T^\beta)}\frac{1}{\log T^\beta}\cdot\frac{q}{T}\sum_{T^\beta \le kq \le T}|r(kq)|\log kq.$$

Since $r(t)\log t \to 0$, we also have

$$\frac{q}{T} \sum_{T^\beta \le kq \le T} |r(kq)| \log kq \to 0$$

as $T \to \infty$, while for the remaining factor in (12.24) we have to use the more precise estimate from (12.22),

$$u^2 = 2 \log T + \frac{2-\alpha}{\alpha} \log \log T + O(1).$$

Since $\delta(t) \to 0$ as $t \to \infty$, we see that for some constant $K > 0$,

$$e^{-u^2(1-\delta(T^\beta)/\log T^\beta)} \le KT^{-2}(\log T)^{-\frac{2-\alpha}{\alpha}}.$$

Thus, since $u^2 \sim 2 \log T$ and $u^{2/\alpha} q \to a$,

$$\frac{T}{q} \sum_{T^\beta \le kq \le T} |r(kq)| e^{-u^2(1-\delta(T^\beta)/\log T^\beta)}$$

$$\le (\frac{T}{q})^2 T^{-2}(\log T)^{-(2-\alpha)/\alpha} \frac{1}{\log T^\beta} o(1)$$

$$= \frac{1}{(u^{2/\alpha} q)^2} (\log T)^{2/\alpha} (\log T)^{-(2-\alpha)/\alpha} \frac{1}{\log T^\beta} o(1) = o(1),$$

and this concludes the proof of the lemma. □

We can now proceed along similar lines as the proof of Theorem 7.6. First, take a fixed $h > 0$, write $n = [T/h]$, and divide $[0,nh]$ into h intervals of length h, and then split each interval into subintervals I_k, I_k^* of length $h-\varepsilon$ and ε, respectively. We then show asymptotic independence of maxima, first giving the following lemmas, corresponding to Lemmas 7.4, 7.5, respectively.

<u>LEMMA 12.13</u> Suppose $u \to \infty$, $q \to 0$, $u^{2/\alpha} q \to a > 0$, (12.1) holds, and $T\mu \to \tau > 0$. Then

(i) $\limsup_{u\to\infty} |P\{M(\bigcup_1^n I_k) \le u\} - P\{M(nh) \le u\}| \le \tau \cdot \frac{\varepsilon}{h}$,

(ii) $\limsup_{u\to\infty} |P\{\xi(jq) \le u,\ jq \in \bigcup_1^n I_k\} - P\{M(\bigcup_1^n I_k) \le u\}| \le \tau\rho_a$,

where $\rho_a \to 0$ as $a \to 0$.

PROOF Part (i) follows at once from Boole's inequality and Theorem 12.9, since

$$0 \le P\{M(\bigcup_1^n I_k) \le u\} - P\{M(nh) \le u\} \le nP\{M(I_1^*) > u\} \sim n\mu\varepsilon \to \tau\cdot\frac{\varepsilon}{h},$$

since $n\mu \sim T\mu/h \to \tau/h$.

Part (ii) follows similarly from Lemma 12.11, which implies

$$0 \le P\{\xi(jq) \le u,\ jq \in \bigcup_1^n I_k\} - P\{M(\bigcup_1^n I_k) \le u\} \le$$

$$\le n \max_k \Big(P\{\xi(jq) \le u,\ jq \in I_k\} - P\{M(I_k) \le u\}\Big)$$

$$\le n\mu(h-\varepsilon)\rho_a + no(\mu) \to \tau(1-\frac{\varepsilon}{h})\rho_a \le \tau\rho_a,$$

where $\rho_a = 1 - H_\alpha(a)/H_\alpha \to 0$ as $a \to 0$. □

LEMMA 12.14 Let $r(t) \to 0$ as $t \to \infty$, and suppose that, as $u^{2/\alpha}q \to a > 0$, (12.23) holds for each $\varepsilon > 0$. Then, as $T \to \infty$, $u^{2/\alpha}q \to a$,

(i) $P\{\xi(jq) \le u,\ jq \in \bigcup_1^n I_k\} - \prod_{k=1}^n P\{\xi(jq) \le u,\ jq \in I_k\} \to 0,$

(ii) $\limsup \left|\prod_{k=1}^n P\{\xi(jq) \le u,\ jq \in I_k\} - P^n\{M(h) \le u\}\right| \le \tau(\rho_a + \frac{\varepsilon}{h})$

where $\rho_a \to 0$ as $a \to 0$.

PROOF The proof of part (i) is identical to that of Lemma 7.5 (i). As for part (ii), we have, by Lemma 12.11,

$$0 \le \prod_{k=1}^n P\{\xi(jq) \le u,\ jq \in I_k\} - \prod_{k=1}^n P\{M(I_k) \le u\}$$

$$\le n \max_k \Big(P\{\xi(jq) \le u,\ jq \in I_k\} - P\{M(I_k) \le u\}\Big)$$

$$\le n\mu(h-\varepsilon)\rho_a + no(\mu)$$

$$\to \tau(1-\frac{\varepsilon}{h})\rho_a \le \tau\rho_a$$

since $n\mu \sim T\mu/h \to \tau/h$. Furthermore, by stationarity

$$0 \leq \prod_{k=1}^{n} P\{M(I_k) \leq u\} - P^n\{M(h) \leq u\} =$$

$$= P^n\{M(I_1) \leq u\} - P^n\{M(h) \leq u\} \leq$$

$$\leq n\Big(P\{M(I_1) \leq u\} - P\{M(h) \leq u\}\Big) \leq$$

$$\leq nP\{M(I_1^*) > u\} \sim n\mu\varepsilon \to \tau \cdot \frac{\varepsilon}{h}. \qquad \square$$

THEOREM 12.15 Let $\{\xi(t)\}$ be a stationary normal process with zero mean and suppose $r(t)$ satisfies (12.1) and (7.2), i.e.

$$r(t) = 1 - C|t|^\alpha + o(|t|^\alpha) \quad \text{as} \quad t \to 0$$

and

$$r(t)\log t \to 0 \quad \text{as} \quad t \to \infty.$$

If $u = u_T \to \infty$ so that $T\mu = TC^{1/\alpha}H_\alpha u^{2/\alpha}\phi(u)/u \to \tau > 0$, then

$$P\{M(T) \leq u\} \to e^{-\tau} \quad \text{as} \quad T \to \infty.$$

PROOF By Lemma 12.12 condition (12.23) of Lemma 12.14 is satisfied, and by Lemmas 12.13 and 12.14 we then have

$$\limsup_{u\to\infty} |P\{M(nh) \leq u\} - P^n\{M(h) \leq u\}| \leq 2\tau(\rho_a + \frac{\varepsilon}{h}),$$

where $\rho_a \to 0$ as $a \to 0$. Letting $\varepsilon \to 0$ and $a \to 0$ this shows that

$$\lim_{n\to\infty} P\{M(nh) \leq u\} - P^n\{M(h) \leq u\} = 0.$$

By Theorem 12.9, $P\{M(h) \leq u\} = 1 - \mu h + o(\mu)$ and hence, as $\mu \sim \tau/T$, $n \sim T/h$,

$$P^n\{M(h) \leq u\} = (1 - \mu h + o(\mu))^n \to e^{-\tau}.$$

Since furthermore

$$M(nh) \leq M(T) \leq M((n+1)h),$$

this proves the theorem. $\square$

As is easily checked, the choice $u_T = \frac{x}{a_T} + b_T$, with a_T and b_T given by (12.2), satisfies $T\mu \to \tau = e^{-x}$, cf. (12.22), and we immediately have the following theorem.

THEOREM 12.16 Suppose $\{\xi(t)\}$ satisfies the conditions of Theorem 12.15, and that

$$a_T = (2\log T)^{1/2}$$

$$b_T = (2\log T)^{1/2} + \frac{1}{(2\log T)^{1/2}}\left\{\frac{2-\alpha}{\alpha}\log\log T + \log\left(C^{1/\alpha}H_\alpha(2\pi)^{-1/2}\,2^{(2-\alpha)/2\alpha}\right)\right\}.$$

Then $P\{a_T(M(T)-b_T)\le x\} \to e^{-e^{-x}}$ as $T\to\infty$. □

Asymptotic properties of ε-upcrossings

As mentioned on p. 88, the asymptotic Poisson character of upcrossings applies also to non-differentiable normal processes, if one considers ε-upcrossings instead of ordinary upcrossings. To prove this, we need to evaluate the expectation of $N_{\varepsilon,u}(T)$, the number of ε-upcrossings of u by $\xi(t)$, $0\le t\le T$.

LEMMA 12.17 Suppose $r(t)$ satisfies (12.1). Then, with h as in Theorem 12.9,

$$\lim_{u\to\infty}\frac{E(N_{\varepsilon,u}(h))}{hu^{2/\alpha}\phi(u)/u}\ \left(=\lim_{u\to\infty}\frac{E(N_{\varepsilon,u}(1))}{u^{2/\alpha}\phi(u)/u}\right) = C^{1/\alpha}H_\alpha.$$

PROOF Write

$$A = \{\xi(t)>u \text{ for some } t\in(-\varepsilon,0]\},$$

$$B = \{\xi(t)>u \text{ for some } t\in(0,\varepsilon)\}.$$

From Theorem 12.9 we have, for $2\varepsilon\le h$,

$$P(A\cup B)\sim 2\varepsilon\, C^{1/\alpha}H_\alpha u^{2/\alpha-1}\phi(u) \quad\text{as } u\to\infty,$$

$$P(A)\sim \varepsilon\, C^{1/\alpha}H_\alpha u^{2/\alpha-1}\phi(u) \quad\text{as } u\to\infty,$$

$$P(B)\sim \varepsilon\, C^{1/\alpha}H_\alpha u^{2/\alpha-1}\phi(u) \quad\text{as } u\to\infty.$$

Hence

$$P(BA^c) = P(A \cup B) - P(A) \sim \varepsilon\, C^{1/\alpha} H_\alpha u^{2/\alpha - 1} \phi(u),$$

and

$$P(BA^c) \le P\{N_{\varepsilon,u}(\varepsilon) = 1\} = E(N_{\varepsilon,u}(\varepsilon)) \le P(B),$$

since there is at most one ε-upcrossing in $[0,\varepsilon]$. Hence

$$E(N_{\varepsilon,u}(\varepsilon)) \sim \varepsilon\, C^{1/\alpha} H_\alpha u^{2/\alpha - 1} \phi(u)$$

and thus

$$E(N_{\varepsilon,u}(1)) = \frac{1}{\varepsilon} E(N_{\varepsilon,u}(\varepsilon)) \sim C^{1/\alpha} H_\alpha u^{2/\alpha - 1} \phi(u)$$

as required. □

In particular the lemma implies that asymptotically the mean number of ε-upcrossings of a suitably increasing level is independent of the choice of $\varepsilon > 0$, and this leads us directly to the Poisson result for the time-normalized number of ε-upcrossings. Let N_T be the point process on $(0,\infty)$ defined by

$$N_T(B) = N_{\varepsilon,u}(T \cdot B),$$

where the level u is chosen so that $T\mu = TC^{1/\alpha} H_\alpha u^{2/\alpha} \phi(u)/u \sim \tau > 0$, and let N be a Poisson process with intensity τ.

<u>THEOREM 12.18</u> Suppose that the assumptions of Theorem 12.15 are satisfied. Then the time-normalized point process N_T of ε-upcrossings of the level u converges in distribution to N as $u \to \infty$, where N is a Poisson process with intensity τ.

<u>PROOF</u> As in the proof of Theorem 8.2 we only have to check that for $c < d$

(a) $\lim_{T\to\infty} E(N_T(c,d]) = E(N(c,d]) = \tau(d-c)$,

and if $R_i = (c_i, d_i]$ (disjoint), $U = \bigcup_{i=1}^{m} R_i$, then

(b) $P\{N_T(U) = 0\} \to P\{N(U) = 0\} = \prod_{i=1}^{m} e^{-\tau|R_i|}$.

By Lemma 12.17, $E(N_T(c,d]) = E(N_{\varepsilon,u}(Tc,Td]) = TE(N_{\varepsilon,u}(c,d]) \sim T(d-c)\mu \sim \tau(d-c)$, which proves (a). For part (b) go through the same steps as in the proof of Theorem 8.2, with only obvious changes. □

In previous chapters we have encountered a variety of results, related to the Poisson convergence of upcrossings of an increasing level. There are no further difficulties in extending these results to cover ε-upcrossings. However, we do not want to lengthen an already long journey over an ocean of lemmas. We just finish by mentioning that a little further generality may be obtained throughout by including a function of slow growth (or perhaps slow decrease) instead of C in (12.1). This has been considered by Qualls and Watanabe (1972), and also by Berman (1971 b).

CHAPTER 13

EXTREMES OF CONTINUOUS PARAMETER STATIONARY PROCESSES

Our primary task in this chapter will be to discuss continuous parameter analogues of the sequence results of Chapter 2, and in particular to obtain a corresponding version of Gnedenko's Theorem which applies in the continuous parameter case.

Specifically throughout this chapter we consider a (strictly) stationary process $\{\xi(t);\ t \geq 0\}$ satisfying the general assumption stated at the start of Chapter 6. In particular it will be assumed that $\xi(t)$ has a.s. continuous sample functions, continuous one-dimensional distributions, and that the underlying probability space is complete. As shown in Lemma 6.1, it then follows that $M(I) = \sup\{\xi(t);\ t \in I\}$ is a r.v. for any interval I and, in particular, so is $M(T) = M([0,T])$.

Our main interest here concerns distributional properties of $M(T)$ as $T \to \infty$, a subject considered for the special (important) case of *normal* processes in Chapters 7 and 12. We shall obtain asymptotic results for the general stationary processes considered here, along the same lines as those for stationary sequences, obtained in Chapter 2. In particular continuous parameter versions of Gnedenko's Theorem will be proved under appropriate dependence restrictions on $\xi(t)$, analogous to the Condition $D(u_n)$. We shall also obtain results of the type $P\{M(T) \leq u_T\} \to e^{-\tau}$ (cf. Theorem 2.6) where $u_T \to \infty$ in an appropriate manner as $T \to \infty$, using a continuous parameter analogue of the Condition $D'(u_n)$.

The general theory will not require that the mean number of upcrossings of a level u be finite, and therefore will include normal processes of the type considered in Chapter 12. As we shall indicate, the Chapter 12 results can be obtained from the general theory of this chapter, though of course the same ultimate amount of work is involved. We shall also consider the special case in which the mean number of upcrossings of any level u per unit time is finite and obtain Poisson limit theorems generalizing those of Chapter 8 to include non-normal processes.

As indicated in Chapter 7, it is convenient to relate the maximum $M(T)$ of the continuous process to the maximum of n terms of a sequence of "submaxima". Specifically if for some conveniently chosen $h > 0$ we write

(13.1) $\zeta_i = \sup\{\xi(t);\ (i-1)h \le t \le ih\}$

then for any $n = 1, 2, \ldots$ we have

(13.2) $M(nh) = \max(\zeta_1, \zeta_2, \ldots, \zeta_n)$.

It is apparent that the properties of $M(T)$ as $T \to \infty$ may be obtained from those of $M(nh)$ by writing $n = [T/h]$ and thus approximating T by nh.

As noted above, we shall consider continuous parameter analogues (to be called $C(u_T)$, $C'(u_T)$) of the Conditions $D(u_n)$, $D'(u_n)$ used for sequences. The Condition $C(u_T)$ will be used in ensuring that the stationary sequence $\{\zeta_n\}$ defined by (13.1) satisfies $D(u_n)$. However before introducing this condition we note a preliminary form of Gnedenko's Theorem which simply *assumes* that the sequence $\{\zeta_n\}$ satisfies $D(u_n)$. This result follows immediately from the sequence case and clearly illustrates the central ideas required in the continuous parameter context. The more complete version (Theorem 13.5) to be given later, of course simply requires finding appropriate conditions, of which the main one will be $C(u_T)$, on $\xi(t)$, to guarantee that $\{\zeta_n\}$ will satisfy $D(u_n)$.

<u>THEOREM 13.1</u> Suppose that for some families of constants $\{a_T > 0\}$, $\{b_T\}$ we have

(13.3) $P\{a_T(M(T) - b_T) \le x\} \xrightarrow{w} G(x)$ as $T \to \infty$

for some non-degenerate G, and that the $\{\zeta_i\}$ sequence defined by (13.1) satisfies $D(u_n)$ whenever $u_n = x/a_{nh} + b_{nh}$ for some fixed $h > 0$ and all real x. Then G is one of the three extreme value types.

PROOF Since (13.3) holds in particular as $T \to \infty$ through values nh and the ζ_n-sequence is clearly stationary, the result follows by replacing ξ_n by ζ_n in Theorem 2.4 and using (13.2). □

Although we shall not make further use of the fact, it is interesting to note that this at once implies that Gnedenko's Theorem holds under "strong mixing" assumptions as the following corollary shows.

COROLLARY 13.2 Theorem 13.1 holds in particular if the $D(u_n)$ conditions are replaced by the assumptions that $\{\xi(t)\}$ is strongly mixing. For then the sequence $\{\zeta_n\}$ is strongly mixing and hence satisfies $D(u_n)$. □

We now introduce the continuous analog of the Condition $D(u_n)$, stated in terms of the finite-dimensional distribution functions $F_{t_1 \ldots t_n}$ of $\xi(t)$, again writing $F_{t_1 \ldots t_n}(u)$ for $F_{t_1 \ldots t_n}(u, \ldots, u)$. The points t_i will be members of a discrete set $\{jq_T;\ j = 1, 2, 3, \ldots\}$ where $\{q_T\}$ is a family of constants tending to zero as $T \to \infty$ at a rate to be specified later.

$C(u_T)$: *The Condition* $C(u_T)$ *will be said to hold for the process* $\xi(t)$ *and the family of constants* $\{u_T;\ T > 0\}$, *with respect to the constants* $q_T \to 0$, *if for any points* $s_1 < s_2 < \ldots < s_p < t_1 < \ldots < t_{p'}$ *belonging to* $\{kq_T;\ 0 \le kq_T \le T\}$ *and satisfying* $t_1 - s_p \ge \gamma$, *we have*

$$(13.4)\quad |F_{s_1 \ldots s_p,\, t_1 \ldots t_{p'}}(u_T) - F_{s_1 \ldots s_p}(u_T)\, F_{t_1 \ldots t_{p'}}(u_T)| \le \alpha_{T,\gamma}$$

where $\alpha_{T,\gamma_T} \to 0$ *for some family* $\gamma_T = o(T)$.

As in the discrete case we may (and do) take $\alpha_{T,\gamma}$ to be non-increasing as γ increases and also note that the condition $\alpha_{T,\gamma_T} \to 0$ for some $\gamma_T = o(T)$ may be replaced by

$$(13.5)\quad \alpha_{T,\lambda T} \to 0 \quad \text{as} \quad T \to \infty$$

for each fixed $\lambda > 0$.

The $D(u_n)$ condition for $\{\xi_n\}$ required in Theorem 13.1 will now be related to $C(u_T)$ by approximating crossings and extremes of the continuous parameter process, by corresponding quantities for a "sampled version". To achieve the approximation we require two conditions involving the maximum of $\xi(t)$ in fixed and in very small time intervals. These conditions are given here in a form which applies very generally – readily verifiable sufficient conditions for important cases are given later in this chapter.

It will be convenient to introduce a function $\psi(u)$ which will generally describe the form of the tail of the distribution of the maximum $M(h)$ in a fixed interval $(0,h)$ as u becomes large. Specifically we shall as needed make one or more of the following successively stronger assumptions:

(13.6) $P\{\xi(0) > u\} = o(\psi(u))$,

(13.7) $P\{M(q) > u\} = o(\psi(u))$ for any $q = q(u) \to 0$ as $u \to \infty$,

(13.8) there exists $h_0 > 0$ such that

$$\limsup_{u\to\infty} P\{M(h) > u\}/(h\psi(u)) \leq 1, \text{ for } 0 < h \leq h_0,$$

(13.9) $P\{M(h) > u\} \sim h\psi(u)$ as $u \to \infty$, for $0 < h \leq h_0$.

Note that Equation (13.9) commonly holds and specifies that the tail of the distribution of $M(h)$ is asymptotically proportional to $\psi(u)$, whereas (13.8) is a weaker assumption which is sometimes convenient as a sufficient condition for the yet weaker (13.7) and (13.6). As we shall see later $\psi(u)$ can also be identified with the mean number of upcrossings of the level u per unit time, $\mu(u)$, in important cases when this is finite. In any case it is of course possible to define $\psi(u)$ to be $P\{M(h_0) > u\}/h_0$ for some fixed $h_0 > 0$, or some asymptotically equivalent function and then attempt to verify any of the above conditions which may be needed.

We shall also require an assumption relating "continuous and discrete" maxima in fixed intervals. Specifically we assume, as required, that for each $a > 0$ there is a family of constants $\{q\} = \{q_a(u)\}$ tending to zero as $u \to \infty$ for each $a > 0$, such that for any fixed $h > 0$,

$$(13.10)\quad \limsup_{u\to\infty} P\{M(h) > u,\ \xi(jq) \le u,\ 0 \le jq \le h\}/\psi(u) \to 0 \quad \text{as} \quad a \to 0.$$

Finally a condition which is sometimes helpful in verifying (13.10) is

$$(13.11)\quad \limsup_{u\to\infty} \frac{P\{\xi(0) \le u,\ \xi(q) \le u,\ M(q) > u\}}{q\,\psi(u)} \to 0 \quad \text{as} \quad a \to 0.$$

Here the constant a specifies the rate of convergence to zero of $q_a(u)$ – as a decreases, the grid of points $\{q_a(u)\}$ tends to become (asymptotically) finer, and for small a the maximum of $\xi(t)$ on the discrete grid approximates the continuous maximum well, as will be seen below. (Simpler versions of (13.10) and (13.11) would be to assume the existence of *one* family $q = q(u)$ of constants such that the upper limits in (13.10) and (13.11) are zero for this family. It can be seen that one can do this without loss of generality in the theorems below, but it seems that, as was the case in Chapter 12, the conditions involving the parameter a often may be easier to check.)

The following lemma contains some simple but useful relationships.

LEMMA 13.3 (i) If (13.8) holds so does (13.7) which in turn implies (13.6). Hence (13.9) clearly implies (13.8), (13.7), and (13.6).

(ii) If I is any interval of length h and (13.6), (13.10) both hold, then there are constants λ_a such that

$$(13.12)\quad 0 \le \limsup_{u\to\infty}[P\{\xi(jq) \le u,\ jq \in I\} - P\{M(I) \le u\}]/\psi(u) \le \lambda_a \to 0$$

as $a \to 0$, where $q = q_a(u)$ is as in (13.10), the convergence being uniform in all intervals of this fixed length h.

(iii) If (13.7) and (13.11) hold so does (13.10) and hence, by (ii) so does (13.12).

(iv) If (13.9) holds and $I_1 = (0,h)$, $I_2 = (h,2h)$ with $0 < h \le h_0/2$, then $P\{M(I_1) > u, M(I_2) > u\} = o(\psi(u))$ as $u \to \infty$.

<u>PROOF</u> (i) If (13.8) holds and $q \to 0$ as $u \to \infty$, then for any fixed $h > 0$, q is eventually smaller than h and $P\{M(q) > u\} \le P\{M(h) > u\}$, so that

$$\limsup_{u\to\infty} P\{M(q) > u\}/\psi(u) \le \limsup_{u\to\infty} P\{M(h) > u\}/\psi(u) \le h$$

by (13.8). Since h is arbitrary it follows that $P\{M(q) > u\}/\psi(u) \to 0$, giving (13.7). It is clear that (13.7) implies (13.6) since $P\{\xi(0) > u\} \le P\{M(q) > u\}$, which proves (i).

To prove (ii) we assume that (13.6) and (13.10) hold and let I be an interval of fixed length h. Since the numbers of points jq in I and in $[0,h]$ differ by at most 2, it is readily seen from stationarity that

$$P\{\xi(jq) \le u,\ jq \in I\} \le P\{\xi(jq) \le u,\ 0 \le jq \le h\} + P\{\xi(0) > u\} + P\{\xi(h) > u\}$$

so that

$$\begin{aligned} 0 &\le P\{\xi(jq) \le u,\ jq \in I\} - P\{M(I) \le u\} \\ &\le P\{\xi(jq) \le u,\ 0 \le jq \le h\} - P\{M(h) \le u\} + 2P\{\xi(0) > u\} \\ &= P\{M(h) > u,\ \xi(jq) \le u,\ 0 \le jq \le h\} + 2P\{\xi(0) > u\} \end{aligned}$$

from which (13.12) follows at once by (13.6) and (13.10), so that (ii) follows.

To prove (iii) we note that there are at most $[h/q]$ complete intervals $[(j-1)q, jq)$ in $[0,h]$ with perhaps a smaller interval remaining so that

$$\begin{aligned} P\{M(h) > u,\ \xi(jq) \le u,\ 0 \le jq \le h\} \le \frac{h}{q} P\{\xi(0) \le u,\ \xi(q) \le u,\ M(q) > u\} \\ + P\{M(q) > u\} \end{aligned}$$

so that (13.10) easily follows from (13.11) and (13.7).

Finally if (13.9) holds and $I_1 = (0,h)$, $I_2 = (h,2h)$ with $0 < h \le h_0/2$, then

$$P\{M(I_2) > u\} = P\{M(I_1) > u\} = h\,\psi(u)(1 + o(1))$$

and

$$P(\{M(I_1) > u\} \cup \{M(I_2) > u\}) = P\{M(I_1 \cup I_2) > u\} = 2h\psi(u)(1+o(1))$$

so that

$$P\{M(I_1) > u,\ M(I_2) > u\} = P\{M(I_1) > u\} + P\{M(I_2) > u\}$$
$$- P(\{M(I_1) > u\} \cup \{M(I_2) > u\})$$
$$= o(\psi(u))$$

as required. □

For $h > 0$, let $\{T_n\}$ be a sequence of time points such $T_n \in [nh, (n+1)h)$ and write $v_n = u_{T_n}$. It is then relatively easy to relate $D(v_n)$ for the sequence $\{\zeta_n\}$ to the condition $C(u_T)$ for the process $\xi(t)$, as the following lemma shows.

<u>LEMMA 13.4</u> Suppose that (13.6) holds for some function $\psi(u)$ and let $\{q_a(u)\}$ be a family of constants for each $a > 0$ with $q_a(u) > 0$, $q_a(u) \to 0$ as $u \to \infty$, and such that (13.10) holds. If $C(u_T)$ is satisfied with respect to the family $q_T = q_a(u_T)$ for each $a > 0$, and $T\psi(u_T)$ is bounded, then the sequence $\{\zeta_n\}$ defined by (13.1) satisfies $D(v_n)$, where $v_n = u_{T_n}$ is as above.

<u>PROOF</u> For a given n, let $i_1 < i_2 < \dots < i_p < j_1 < \dots < j_{p'} < n$, $j_1 - i_p \geq \ell$. Write $I_r = [(i_r - 1)h,\ i_r h]$, $J_s = [(j_s - 1)h,\ j_s h]$. For brevity write q for the elements in one of the families $\{q_a(\cdot)\}$ and let

$$A_q = \bigcap_{r=1}^{p} \{\xi(jq) \leq v_n,\ jq \in I_r\}, \qquad A = \bigcap_{r=1}^{p} \{\zeta_{i_r} \leq v_n\}$$

$$B_q = \bigcap_{s=1}^{p'} \{\xi(jq) \leq v_n,\ jq \in J_s\}, \qquad B = \bigcap_{s=1}^{p'} \{\zeta_{j_s} \leq v_n\}.$$

It follows in an obvious way from Lemma 13.3 (ii) that

$$0 \leq \limsup_{n\to\infty} \{P(A_q B_q) - P(AB)\} \leq \limsup_{n\to\infty} (p + p')\psi(v_n)\lambda_a$$

$$\leq \limsup_{n\to\infty} n\psi(v_n)\lambda_a \leq K\lambda_a$$

for some constant K (since $nh \sim T_n$ and $T_n\psi(v_n)$ is bounded) and where $\lambda_a \to 0$

as $a \to 0$. Similarly

$$\limsup_{n\to\infty} |P(A_q) - P(A)| \le K\lambda_a, \quad \limsup_{n\to\infty} |P(B_q) - P(B)| \le K\lambda_a.$$

Now

$$\begin{aligned} |P(A \cap B) - P(A)P(B)| &\le |P(A \cap B) - P(A_q \cap B_q)| \\ &\quad + |P(A_q \cap B_q) - P(A_q)P(B_q)| + P(A_q)|P(B_q) - P(B)| \\ &\quad + P(B)|P(A_q) - P(A)| \\ &= R_{n,a} + |P(A_q \cap B_q) - P(A_q)P(B_q)| \end{aligned} \tag{13.13}$$

where $\limsup_{n\to\infty} R_{n,a} \le 3K\lambda_a$.

Since the largest jq in any I_r is at most $i_p h$, and the smallest in any J_s is at least $(j_1 - 1)h$, their difference is at least $(\ell - 1)h$. Also the largest jq in $J_{p'}$ does not exceed $j_{p'}h \le nh \le T_n$ so that from (13.4) and (13.13),

$$|P(A \cap B) - P(A)P(B)| \le R_{n,a} + \alpha^{(a)}_{T_n,(\ell-1)h}, \tag{13.14}$$

in which the dependence of $\alpha_{T,\ell}$ on a is explicitly indicated. Write now $\alpha^*_{n,\ell} = \inf_{a>0}\{R_{n,a} + \alpha^{(a)}_{T_n,(\ell-1)h}\}$. Since the left-hand side of (13.14) does not depend on a we have

$$|P(A \cap B) - P(A)P(B)| \le \alpha^*_{n,\ell},$$

which is precisely the desired conclusion of the lemma, provided we can show that $\lim_{n\to\infty} \alpha^*_{n,\lambda n} = 0$ for any $\lambda > 0$ (cf. (2.3)). But, for any $a > 0$,

$$\alpha^*_{n,\lambda n} \le R_{n,a} + \alpha^{(a)}_{T_n,(\lambda n-1)h} \le R_{n,a} + \alpha^{(a)}_{T_n,\lambda T_n/2}$$

when n is sufficiently large (since $\alpha^{(a)}_{T,\ell}$ decreases in ℓ), and hence by (13.5)

$$\limsup_{n\to\infty} \alpha^*_{n,\lambda n} \le 3K\lambda_a,$$

and since a is arbitrary and $\lambda_a \to 0$ as $a \to 0$, it follows that $\alpha^*_{n,\lambda n} \to 0$ as desired. □

The general continuous version of Gnedenko's Theorem is now readily restated in terms of conditions on $\xi(t)$ itself.

THEOREM 13.5 With the above notation for the stationary process $\{\xi(t)\}$ satisfying (13.6) for some function ψ suppose that there are constants $a_T > 0$, b_T such that

$$P\{a_T(M(T) - b_T) \leq x\} \to G(x)$$

for a non-degenerate G. Suppose that $T\psi(u_T)$ is bounded and $C(u_T)$ holds for $u_T = x/a_T + b_T$ for each real x, with respect to families of constants $\{q_a(u)\}$ satisfying (13.10). Then G is one of the three extreme value distributional types.

PROOF This follows at once from Theorem 13.1 and Lemma 13.4, by choosing $T_n = nh$. □

As noted the conditions of this theorem are of a general kind, and more specific sufficient conditions will be given in the applications later in this chapter.

Convergence of $P\{M(T) \leq u_T\}$

Gnedenko's Theorem involved consideration of $P\{a_T\{M(T) - b_T\} \leq x\}$, which may be rewritten as $P\{M(T) \leq u_T\}$ with $u_T = x/a_T + b_T$. We turn now to the question of convergence of $P\{M(T) \leq u_T\}$ as $T \to \infty$ for families u_T which are not necessarily linear functions of a parameter x. (This is analogous to the convergence of $P(M_n \leq u_n)$ for sequences, of course.) These results are of interest in their own right, but also since they make it possible to simply modify the classical criteria for domains of attraction to the three limiting distributions, to apply in this continuous parameter context.

Our main purpose is to demonstrate the equivalence of the relations

$P\{M(h) > u_T\} \sim \tau/T$ and $P\{M(T) \le u_T\} \to e^{-\tau}$ under appropriate conditions. The following condition will be referred to as $C'(u_T)$ and is analogous to $D'(u_n)$ defined in Chapter 2, for sequences.

<u>$C'(u_T)$</u>: *The Condition* $C'(u_T)$ *will be said to hold for the process* $\{\xi(t)\}$ *and the family of constants* $\{u_T;\ T > 0\}$, *with respect to the constants* $q_T \to 0$ *if* $\limsup_{T\to\infty} \frac{T}{q} \sum_{h<jq<\varepsilon T} P\{\xi(0) > u_T,\ \xi(jq) > u_T\} \to 0$ as $\varepsilon \to 0$, *for some* $h > 0$.

The following lemma will be useful in obtaining the desired equivalence.

<u>LEMMA 13.6</u> Suppose that (13.9) holds for some function ψ, and let $\{u_T\}$ be a family of levels such that $C'(u_T)$ holds with respect to families $\{q_a(u)\}$ satisfying (13.10), for each $a > 0$, with h in $C'(u_T)$ not exceeding $h_0/2$ in (13.9). Then $T\psi(u_T)$ is bounded, and writing $n' = [n/k]$, for n and k integers,

(13.15) $0 \le \limsup_{n\to\infty} [n'P\{M(h) > v_n\} - P\{M(n'h) > v_n\}] = o(k^{-1})$, as $k \to \infty$,

with $v_n = u_{T_n}$, for any sequence $\{T_n\}$ with $T_n \in [nh, (n+1)h)$.

<u>PROOF</u> We shall use the extra assumption

(13.16) $\liminf_{T\to\infty} T\psi(u_T) > 0$,

in proving $T\psi(u_T)$ bounded and (13.15). It is then easily checked (e.g. by replacing $T\psi(u_T)$ be $\max(1, T\psi(u_T))$ in the proof) that the result holds also without the extra assumption.

Now, write $I_j = [(j-1)h, jh]$, $j = 1, 2, \ldots$ and $M_q(I) = \max\{\xi(jq);\ jq \in I\}$, for any interval I. We shall first show that (assuming (13.16) holds)

(13.17) $0 \le \limsup_{n\to\infty} \frac{1}{T_n\psi(v_n)} [n'P\{M(h) > v_n\} - P\{M(n'h) > v_n\}] = o(k^{-1})$

as $k \to \infty$. The expression in (13.17) is clearly non-negative, and by stationarity and the fact that $M \ge M_q$, does not exceed

$$\limsup_{n\to\infty} \frac{1}{T_n\psi(v_n)} \sum_{j=1}^{n'} [P\{M(I_j) > v_n\} - P\{M_q(I_j) > v_n\}]$$

$$+ \limsup_{n\to\infty} \frac{1}{T_n\psi(v_n)} [\sum_{j=1}^{n'} P\{M_q(I_j) > v_n\} - P\{M_q(n'h) > v_n\}].$$

By Lemma 13.3 (ii), the first of the upper limits does not exceed $\lambda_a \limsup_{n\to\infty} n'/T_n = \lambda_a/(hk)$, where $\lambda_a \to 0$ as $a \to 0$. The expression in the second upper limit may be written as

$$\frac{1}{T_n\psi(v_n)} [\sum_{j=1}^{n'} P\{M_q(I_j) > v_n\} - \sum_{j=1}^{n'} P\{M_q(I_j) > v_n,\ M_q(\bigcup_{\ell=j+1}^{n'} I_\ell) \le v_n\}]$$

$$\le \frac{1}{T_n\psi(v_n)} \sum_{j=1}^{n'} P\{M_q(I_j) > v_n,\ M_q(I_{j+1}) > v_n\}$$

$$+ \frac{1}{T_n\psi(v_n)} \sum_{j=1}^{n'} P\{M_q(I_j) > v_n,\ M_q(\bigcup_{\ell=j+2}^{n'} I_\ell) > v_n\}$$

$$\le \frac{n'}{T_n}\, o(1) + \frac{n'h}{qT_n\psi(v_n)} \sum_{h\le jq\le n'h} P\{\xi(0) > v_n,\ \xi(jq) > v_n\},$$

by Lemma 13.3 (iv) and some obvious estimation using stationarity. By $C'(u_T)$, using (13.16), the upper limit (over n) of the last term is readily seen to be $o(k^{-1})$ for each $a > 0$, and (13.17) now follows by gathering these facts.

Further, by (13.17) and (13.9)

$$\liminf_{n\to\infty} \frac{1}{T_n\psi(v_n)} \ge \liminf_{n\to\infty} \frac{1}{T_n\psi(v_n)} P\{M(n'h) > v_n\}$$

$$\ge \liminf_{n\to\infty} \frac{1}{T_n\psi(v_n)} n'P\{M(h) > v_n\}$$

$$- \limsup_{n\to\infty} \frac{1}{T_n\psi(v_n)} [n'P\{M(h) > v_n\} - P\{M(n'h) > v_n\}]$$

$$= \frac{1}{k} - o(\frac{1}{k}),$$

and hence $\liminf_{n\to\infty}(T_n\psi(v_n))^{-1} > 0$. Thus $T_n\psi(u_{T_n})$ is bounded for any sequence $\{T_n\}$ satisfying $nh \le T_n \le (n+1)h$, which readily implies that $T\psi(u_T)$ is bounded. Finally, (13.15) then follows at once from (13.17).

□

COROLLARY 13.7 Under the conditions of the lemma, if $\lambda_{n,k} = |n'h\psi(v_n) - P\{M(n'h) > v_n\}|$, then $\limsup_{n\to\infty} \lambda_{n,k} = o(k^{-1})$ as $k \to \infty$.

PROOF Noting that $n'\psi(v_n)$ is bounded, this follows at once from the lemma, by (13.9). □

Our main result now follows readily.

THEOREM 13.8 Suppose that (13.9) holds for some function ψ, and let $\{u_T\}$ be a family of constants such that for each $a > 0$, $C(u_T)$ and $C'(u_T)$ hold with respect to the family $\{q_a(u)\}$ of constants satisfying (13.10), with h in $C'(u_T)$ not exceeding $h_0/2$ in (13.9). Then

(13.18) $T\psi(u_T) \to \tau > 0$

if and only if

(13.19) $P\{M(T) \leq u_T\} \to e^{-\tau}$.

PROOF If (13.9), (13.10), and $C'(u_T)$ hold as stated, then $T\psi(u_T)$ is bounded according to Lemma 13.6 and by Lemma 13.4 the sequence of "submaxima" $\{\zeta_n\}$ defined by (13.1) satisfies $D(v_n)$, with $v_n = u_{T_n}$, for any sequence $\{T_n\}$ with $T_n \in [nh, (n+1)h)$. Hence from Lemma 2.3, writing $n' = [n/k]$,

(13.20) $P\{M(nh) \leq v_n\} - P^k\{M(n'h) \leq v_n\} \to 0$ as $n \to \infty$.

Clearly it is enough to prove that

(13.21) $T_n\psi(v_n) \to \tau > 0$

if and only if

(13.22) $P\{M(T_n) \leq v_n\} \to e^{-\tau}$,

for any sequence $\{T_n\}$ with $T_n \in [nh, (n+1)h)$. Further, $T\psi(u_T)$ bounded implies that $\psi(u_T) \to 0$ as $T \to \infty$ so that

$$0 \leq P\{M(nh) \leq v_n\} - P\{M(T_n) \leq v_n\}$$
$$\leq P\{M(h) > v_n\} \sim h\psi(v_n) \to 0,$$

and thus (13.22) holds if and only if

(13.23) $P\{M(nh) \le v_n\} \to e^{-\tau}$.

Hence it is sufficient to prove that (13.21) and (13.23) are equivalent under the hypothesis of the theorem.

Suppose now that (13.21) holds so that in particular

(13.24) $n'h\psi(v_n) \to \tau/k$ as $n \to \infty$.

With the notation of Corollary 13.7 we have

(13.25) $1 - n'h\psi(v_n) - \lambda_{nk} \le P\{M(n'h) \le v_n\} \le 1 - n'h\psi(v_n) + \lambda_{nk}$

so that, letting $n \to \infty$,

$$\begin{aligned} 1 - \tau/k - o(k^{-1}) &\le \liminf_{n\to\infty} P\{M(n'h) \le v_n\} \\ &\le \limsup_{n\to\infty} P\{M(n'h) \le v_n\} \\ &\le 1 - \tau/k + o(k^{-1}). \end{aligned}$$

By taking k-th powers throughout and using (13.20) we obtain

$$\begin{aligned} (1 - \tau/k - o(k^{-1}))^k &\le \liminf_{n\to\infty} P\{M(nh) \le v_n\} \\ &\le \limsup_{n\to\infty} P\{M(nh) \le v_n\} \\ &\le (1 - \tau/k + o(k^{-1}))^k, \end{aligned}$$

and letting k tend to infinity proves (13.23).

Hence (13.21) implies (13.23) under the stated conditions. We shall now show that conversely (13.23) implies (13.21). The first part of the above proof still applies so that (13.20) and the conclusion of Corollary 13.7, and hence (13.25), hold. A rearrangement of (13.25) gives

$$\begin{aligned} 1 - P\{M(n'h) \le v_n\} - \lambda_{nk} &\le n'h\psi(v_n) \\ &\le 1 - P\{M(n'h) \le v_n\} + \lambda_{nk}. \end{aligned}$$

But it follows from (13.20) and (13.23) that $P\{M(n'h) \le v_n\} \to e^{-\tau/k}$ and hence, using Corollary 13.7, that

$$1 - e^{-\tau/k} - o(k^{-1}) \leq \liminf_{n\to\infty} n'h\psi(v_n)$$
$$\leq \limsup_{n\to\infty} n'h\psi(v_n)$$
$$\leq 1 - e^{-\tau/k} + o(k^{-1}).$$

Multiplying through by k and letting $k \to \infty$ shows that $T_n\psi(v_n) \sim$ $\sim nh\psi(v_n) \to \tau$, and concludes the proof that (13.23) implies (13.21). □

Associated sequence of independent variables

With a slight change of emphasis from Chapter 2 we say that any i.i.d. sequence $\hat{\zeta}_1, \hat{\zeta}_2, \ldots$ whose marginal d.f. F satisfies

$$1 - F(u) \sim P\{M(h) > u\}$$

for some $h > 0$, is an *independent sequence associated with* $\{\xi(t)\}$. If (13.9) holds this is clearly equivalent to the requirement

(13.26) $1 - F(u) \sim h\psi(u)$ as $u \to \infty$.

Theorem 13.8 may then be related to the corresponding result for i.i.d. sequences in the following way.

THEOREM 13.9 Let $\{u_T\}$ be a family of constants such that the conditions of Theorem 13.8 hold, and let $\hat{\zeta}_1, \hat{\zeta}_2, \ldots$ be an associated independent sequence. Let $0 < \rho < 1$. If

(13.27) $P\{M(T) \leq u_T\} \to \rho$ as $T \to \infty$

then

(13.28) $P\{\hat{M}_n \leq v_n\} \to \rho$ as $n \to \infty$

with $v_n = u_{nh}$. Conversely, if (13.28) holds for some sequence $\{v_n\}$ then (13.27) holds for any $\{u_T\}$ such that $\psi(u_T) \sim \psi(v_{[T/h]})$, provided the conditions of Theorem 13.8 hold.

PROOF If (13.27) holds, and $\rho = e^{-\tau}$, Theorem 13.8 and (13.26) give

$$1 - F(u_{nh}) \sim h\psi(u_{nh}) \sim \tau/n,$$

so that $P\{\hat{M}_n \leq u_{nh}\} \to e^{-\tau}$, giving (13.28). Conversely, (13.28) and (13.26) imply that $h\psi(v_n) \sim 1 - F(v_n) \sim \tau/n$ and hence

$$T\psi(u_T) \sim T\psi(v_{[T/h]}) \sim T\tau/(h[T/h]) \to \tau$$

so that (13.27) holds by Theorem 13.8. □

These results show how the function ψ may be used in the classical criteria for domains of attraction to determine the asymptotic distribution of $M(T)$. We write $\mathcal{D}(G)$ for the domain of attraction to the (extreme value) d.f. G, i.e. the set of all d.f.'s F such that $F^n(x/a_n + b_n) \to G(x)$ for some sequences $\{a_n > 0\}$, $\{b_n\}$.

<u>THEOREM 13.10</u> Suppose that the conditions of Theorem 13.8 hold for all families $\{u_T\}$ of the form $u_T = x/a_T + b_T$, where $a_T > 0$, b_T are given constants, and that

(13.29) $P\{a_T(M(T) - b_T) \leq x\} \to G(x)$.

Then (13.26) holds for some $F \in \mathcal{D}(G)$. Conversely, suppose (13.9) holds and that (13.26) is satisfied for some $F \in \mathcal{D}(G)$, let $a_n' > 0$, b_n' be constants such that $F^n(x/a_n' + b_n') \to G(x)$, and define $a_T = a'_{[T/h]}$, $b_T = b'_{[T/h]}$. Then (13.29) holds, provided the conditions of Theorem 13.8 are satisfied for each $u_T = x/a_T + b_T$, $-\infty < x < \infty$.

<u>PROOF</u> If (13.29) holds, together with the conditions stated, Theorem 13.9 applies, so that in particular

$$P\{a_{nh}(\hat{M}_n - b_{nh}) \leq x\} \to G(x)$$

where $\hat{M}_n$ is the maximum of the independent sequence of associated variables $\hat{\zeta}_1, \ldots, \hat{\zeta}_n$. It follows at once that their marginal d.f. F belongs to $\mathcal{D}(G)$, and (13.26) is immediate by definition.

Conversely, suppose (13.26) holds for some d.f. $F \in \mathcal{D}(G)$, and let $\hat{\zeta}_1, \hat{\zeta}_2, \ldots$ be an i.i.d. sequence with marginal d.f. F, and suppose that for $v_n = x/a_n' + b_n'$,

$$P\{\hat{M}_n \le v_n\} \to G(x) \quad \text{as} \quad n \to \infty.$$

Then clearly, for $a_T = a'_{[T/h]}$, $b_T = b'_{[T/h]}$, and $u_T = x/a_T + b_T$,

$$\psi(u_T) = \psi(v_{[t/h]})$$

so that Theorem 13.9 applies, giving (13.29). □

Stationary normal processes

Although we have obtained the asymptotic distributional properties of the maximum of stationary normal processes directly, it is of interest to see how these may be obtained as applications of the general theory of this chapter. This does not lessen the work involved, of course, since the same calculations in the "direct route" are used to verify the conditions in the general theory. However the use of the general theory does also give insight and perspective regarding the principles involved. We deal here with the more general normal processes considered in Chapter 12. This will include the (Chapter 7) normal processes with finite second spectral moments considered in Chapter 7, of course. The latter processes may also be treated as particular cases of general processes with finite upcrossings intensities – a class dealt with later in this chapter.

Suppose then that $\xi(t)$ is a stationary normal process with zero mean and covariance function (12.1), viz.,

$$(13.30) \quad r(\tau) = 1 - C|\tau|^{\alpha} + o(|\tau|^{\alpha}) \quad \text{as} \quad \tau \to 0$$

where $0 < \alpha \le 2$. The major result to be obtained is Theorem 12.15, restated here.

__THEOREM 12.15__ Let $\{\xi(t)\}$ be a zero-mean stationary normal process, with covariance function $r(t)$ satisfying (13.30) and

(13.31) $r(t)\log t \to 0$, as $t \to \infty$.

If $u = u_T \to \infty$ and $\mu = \mu(u) = C^{1/\alpha} H_\alpha u^{2/\alpha} \psi(u)/u$ (with H_α defined in Theorem 12.9), and if $T\mu(u_T) \to \tau$, then $P\{M(T) \le u\} \to e^{-\tau}$ as $T \to \infty$.

PROOF FROM THE GENERAL THEORY Write $\psi(u) = \mu(u)$ so that $T\psi(u_T) \to \tau$. Theorem 12.9 shows at once that (13.9) holds (for all $h > 0$). Define $q_a(u) = au^{-2/\alpha}$, and note that (13.10) holds, by (12.21). Hence the result will follow at once if $C(u_T)$, $C'(u_T)$ are both shown to hold with respect to $\{q_a(u)\}$ for each $a > 0$.

It is easily seen in a familiar way that $C(u_T)$ holds. For by Lemma 3.2 the left hand side of (13.4) (with $q_a(u)$ for $q(u)$, $u = u_T$) does not exceed

$$K \sum_{i=1}^{p} \sum_{j=1}^{p'} |r(t_j - s_i)| \, e^{-u^2/(1+|r(t_j-s_i)|)}$$

which is dominated by

$$K \frac{T}{q} \sum_{\gamma \le kq \le T} |r(kq)| \, e^{-u^2/(1+|r(kq)|)},$$

and this tends to zero for each $\{q\} = \{q_a\}$, a fixed, by Lemma 12.12. If we identify this expression with $\alpha_{T,\gamma}$ then (13.5) holds almost trivially since $\alpha_{T,\lambda T} \le \alpha_{T,\gamma}$ for any fixed γ when $\lambda T > \gamma$.

$C'(u_T)$ follows equally simply by Lemma 3.2 which gives

$$|P\{\xi(0) > u, \xi(jq) > u\} - (1-\Phi(u))^2| \le K|r(jq)| \, e^{-u^2/(1+|r(jq)|)}$$

so that

$$\frac{T}{q} \sum_{h < jq \le \varepsilon T} P\{\xi(0) > u, \xi(jq) > u\}$$

$$\le \frac{\varepsilon T^2}{q^2} (1-\Phi(u))^2 + K \frac{T}{q} \sum_{h < jq \le \varepsilon T} |r(jq)| \, e^{-u^2/(1+|r(jq)|)} .$$

The second term tends to zero as $T \to \infty$ again by Lemma 12.12. The first term is asymptotically equivalent to

$$\frac{\varepsilon T^2}{q^2} \frac{(\psi(u))^2}{u^2} \to \frac{\varepsilon \tau^2}{a^2 C^{2/\alpha} H_\alpha^2}$$

by the definitions of q and $\psi(u)$, and the fact that $T\psi(u) \to \tau$. Since $\varepsilon\tau^2/a^2 \to 0$ for each fixed a as $\varepsilon \to 0$, $C'(u_T)$ follows. □

Finally, we note that the "double exponential limiting distribution" for the maximum $M(T)$ (Theorem 12.16) follows exactly as before from Theorem 12.15.

Processes with finite upcrossing intensities

We show now how some of the conditions required for the general theory may be simplified when the mean number $\mu(u)$ of upcrossings of each level u per unit time is finite. This includes the particular normal cases with finite second spectral moments already covered in Chapter 7 and in the preceding section, but of course not the "non-differentiable" process with $\alpha < 2$.

We use the notation of Chapter 6 in addition to that of the present chapter and assume that $\mu = \mu(u) = E(N_u(1)) < \infty$ for each value of u. Writing as in (6.1), for $q > 0$,

(13.32) $J_q(u) = P\{\xi(0) < u < \xi(q)\}/q$

it is clear that

(13.33) $J_q(u) \leq P\{N_u(q) \geq 1\}/q \leq E(N_u(q))/q = \mu$

and it follows from Lemma 6.3 that

(13.34) $J_q(u) \to \mu$ as $q \to 0$

for each fixed u.

In the *normal* case we saw (Lemma 6.6) that $J_q(u) \sim \mu(u)$ as $q \to 0$

in such a way that $uq \to 0$. Here we shall use a variant of this property assuming as needed that, for each $a > 0$, there are constants $q_a(u) \to 0$ as $u \to \infty$ such that, with $q_a = q_a(u)$, $\mu = \mu(u)$,

(13.35) $\liminf_{u\to\infty} J_{q_a}(u)/\mu \geq \nu_a$

where $\nu_a \to 1$ as $a \to 0$. (As indicated below this is readily verified when $\xi(t)$ is normal when we may take $q_a(u) = a/u$.)

We shall assume as needed that

(13.36) $P\{\xi(0) > u\} = o(\mu(u))$ as $u \to \infty$,

which clearly holds for the normal case but more generally is readily verified if, for example for some $q = q(u) \to 0$ as $u \to \infty$

(13.37) $\limsup_{u\to\infty} \dfrac{P\{\xi(0) > u,\ \xi(q) > u\}}{P\{\xi(0) > u\}} < 1,$

since (13.37) implies that $\liminf_{u\to\infty} qJ_q(u)/P\{\xi(0) > u\} > 0$, from which it follows that $P\{\xi(0) > u\}/J_q(u) \to 0$, and hence (13.36) holds since $J_q(u) \leq \mu(u)$.

We may now recast the conditions (13.8) and (13.9) in terms of the function $\mu(u)$, identifying this function with $\psi(u)$.

LEMMA 13.11 (i) Suppose $\mu(u) < \infty$ for each u and that (13.36) (or the sufficient condition (13.37)) holds. Then (13.8) holds with $\psi(u) = \mu(u)$.

(ii) If (13.35) holds for some family $\{q_a(u)\}$ then (13.11) holds with $\psi(u) = \mu(u)$.

PROOF Since clearly

$$P\{M(h) > u\} \leq P\{N_u(h) \geq 1\} + P\{\xi(0) > u\} \leq \mu h + P\{\xi(0) > u\},$$

(13.8) follows at once from (13.36), which proves (i).

Now, if (13.35) holds, then with $q = q_a(u)$, $\mu = \mu(u)$,

$$\begin{aligned} P\{\xi(0) \le u,\ \xi(q) \le u,\ M(q) > u\} &= \\ &= P\{\xi(0) \le u,\ M(q) > u\} - P\{\xi(0) \le u < \xi(q)\} \\ &\le P\{N_u(q) \ge 1\} - qJ_q(u) \\ &\le \mu q - \mu q \nu_a (1 + o(1)) \end{aligned}$$

so that

$$\limsup_{u\to\infty} P\{\xi(0) \le u,\ \xi(q) \le u,\ M(q) > u\}/(q\mu) \le 1 - \nu_a ,$$

which tends to zero as $a \to 0$, giving (13.11). □

In view of this lemma, Gnedenko's Theorem now applies to processes of this kind using the more readily verifiable conditions (13.35) and (13.36), as follows.

THEOREM 13.12 Theorem 13.5 holds for a stationary process $\{\xi(t)\}$ with $\psi(u) = \mu(u) < \infty$ for each u if the conditions (13.6) and (13.10) are replaced by (13.35) and (13.36) (or by (13.35) and (13.37)).

PROOF By (i) of the previous lemma the condition (13.36) (or its sufficient condition (13.37)) implies (13.8) and hence both (13.6) and (13.7). On the other hand (ii) of the lemma shows that (13.35) implies (13.11) which together with (13.7) implies (13.10) by Lemma 13.3 (iii). □

The condition (13.10) also occurs in Theorem 13.8 and may of course be replaced by (13.35) there, since (13.7) is implied by (13.9) which is assumed in that theorem.

Finally we note that while (13.36) and (13.37) are especially convenient to give (13.8) (Lemma 13.11 (i)), the verification of (13.9) still requires obtaining

$$\liminf_{u\to\infty} P\{M(h) > u\}/(h\mu(u)) \ge 1 \quad \text{for} \quad 0 \le h \le h_0.$$

This of course follows for all normal processes considered by Theorem 12.9, with $\alpha = 2$. There are a number of independent simpler derivations of this when $\alpha = 2$, one of these being along the lines of the "cosine-process" comparison in Chapter 7. The actual comparison used there gave a slightly weaker result, which was, however, sufficient to yield the desired limit theory by the particular methods employed.

Poisson convergence of upcrossings

It was shown in Chapter 8 that the upcrossings of one or more high levels by a normal process $\xi(t)$ take on a specific Poisson character under appropriate conditions. It was assumed in particular that the covariance function $r(t)$ of $\xi(t)$ satisfied (7.1) so that the expected number of upcrossings per unit time, $\mu = E(N_u(1))$, is finite.

Corresponding results are obtainable for ε-upcrossings by normal processes when $r(t)$ satisfies (12.1) with $\alpha < 2$ and indeed the proof is indicated in Chapter 12 for the single level result (Theorem 12.18).

For general stationary processes the same results may be proved under conditions used in this present chapter, including C, C'. Again when $\mu = E(N_u(1)) < \infty$ the results apply to actual upcrossings while if $\mu = \infty$ they apply to ε-upcrossings. We shall state and briefly indicate the proof of the specific theorem for a single level in the case when $\mu < \infty$.

As in previous discussions, we consider a time period T and a level u_T both increasing in such a way that $T\mu \to \tau > 0 \; (\mu = \mu(u_T))$, and define a normalized point process of upcrossings by

$$N_T^*(B) = N_{u_T}(TB), \quad (N_T^*(t) = N_{u_T}(tT))$$

for each interval (or more general Borel set) B, so that, in particular,

$$E(N_T^*(1)) = E(N_{u_T}(t)) = \mu T \to \tau.$$

This shows that the "intensity" (i.e. mean number of events per unit time) of the normalized upcrossing point process converges to τ. Our task is to show that the upcrossing point process actually converges in distribution to a Poisson process with mean τ.

The derivation of this result is based on the following two extensions of Theorem 13.8 which are proved by similar arguments to those used in obtaining Theorem 13.8 and in Chapter 8.

THEOREM 13.13 Under the conditions of Theorem 13.8, if $0 < \theta < 1$ and $\mu T \to \tau$, then

(13.38) $P\{M(\theta T) \le u_T\} \to e^{-\theta\tau}$ as $T \to \infty$.

THEOREM 13.14 If $I_1, I_2, \ldots, I_r$ are disjoint subintervals of $[0,1]$ and $I_j^* = TI_j = \{t;\ t/T \in I_j\}$ then under the conditions of Theorem 13.8 if $\mu T \to \tau$,

(13.39) $$P\left(\bigcap_{j=1}^{r} \{M(I_j^*) \le u_T\}\right) - \prod_{j=1}^{r} P\{M(I_j^*) \le u_T\} \to 0,$$

so that by Theorem 13.13

(13.40) $$P\left(\bigcap_{j=1}^{r} \{M(I_j^*) \le u_T\}\right) \to e^{-\tau \sum_{j=1}^{r} \theta_j},$$

where θ_j is the length of I_j, $1 \le j \le r$.

It is now a relatively straightforward matter to show that the point processes N_T^* converge (in the full sense of weak convergence) to a Poisson process N with intensity τ.

THEOREM 13.15 Under the conditions of Theorem 13.8 if $T\mu \to \cdot$ where $\mu = \mu(u_T)$, then the family N_T^* of normalized point processes of upcrossings of u_T on the unit interval converges in distribution to a Poisson process N with intensity τ on that interval as $T \to \infty$.

PROOF Again by Theorem A.1 it is sufficient to prove that

(i) $E(N_T^*\{(a,b]\}) \to E(N\{(a,b]\}) = \tau(b-a)$ as $T \to \infty$ for all a,b, $0 \le a \le b \le 1$.

(ii) $P\{N_T^*(B)=0\} \to P\{N(B)=0\}$ as $T\to\infty$ for all sets B of the form $\bigcup_1^r B_j$ where r is any integer and B_j are disjoint intervals $(a_j,b_j] \subset (0,1]$.

Now (i) follows trivially since

$$E(N_T^*\{(a,b]\}) = \mu T(b-a) \to \tau(b-a).$$

To obtain (ii) we note that

$$0 \le P\{N_T^*(B)=0\} - P\{M(TB) \le u_T\}$$

$$= P\{N_u(TB) = 0,\ M(TB) > u_T\}$$

$$\le \sum_{j=1}^{r} P\{\xi(Ta_j) > u_T\}$$

since if the maximum in $TB = \bigcup_{j=1}^{r} (Ta_j,\ Tb_j]$ exceeds u_T, but there are no upcrossings of u_T in these intervals, then $\xi(t)$ must exceed u_T at the initial point of at least one such interval. But the last expression is just $rP\{\xi(0) > u_T\} \to 0$ as $T\to\infty$. Hence

$$P\{N_T^*(B)=0\} - P\{M(TB) \le u_T\} \to 0.$$

But $P\{M(TB) \le u_T\} = P\{\bigcap_{j=1}^{r} (M(TB_j) \le u_T\} \to e^{-\tau\Sigma(b_j-a_j)}$ by Theorem 13.14 so that (ii) follows since $P\{N(B)=0\} = e^{-\tau\Sigma(b_j-a_j)}$. □

<u>COROLLARY 13.16</u> If B_j are disjoint (Borel) subsets of the unit interval and if the boundary of each B_j has zero Lebesgue measure then

$$P\{N_T^*(B_j) = r_j,\ 1 \le j \le n\} \to \prod_{j=1}^{r} e^{-\tau m(B_j)} \frac{[\tau m(B_j)]^{r_i}}{r_j!}$$

where $m(B_j)$ denotes the Lebesgue measure of B_j.

<u>PROOF</u> This is an immediate consequence of the full distributional convergence proved (cf. Appendix). □

The above results concern convergence of the point processes of upcrossings of u_T in the unit interval to a Poisson process in the unit interval. A slight modification, requiring C and C' to hold for all families $u_{\theta T}$ in place of u_T for all $\theta > 0$, enables a corresponding result to be shown for the upcrossings on the whole positive real line, but we do not pursue this here. Instead we show how Theorem 13.15 yields the asymptotic distribution of the r-th largest local maximum in $(0,T)$.

Suppose, then, that $\xi(t)$ has a continuous derivative a.s. and (cf. Chapters 6 and 9) define $N_u'(T)$ to be the number of local maxima in the interval $(0,T)$ for which the process value exceeds u, i.e. the number of downcrossing points t of zero by ξ' in $(0,T)$ such that $\xi(t) > u$. Clearly $N_u'(T) \geq N_u(T) - 1$ since at least one local maximum occurs between two upcrossings. It is also reasonable to expect that if the sample function behaviour is not too irregular, there will tend to be just one local maximum above u between most successive upcrossings of u when u is large, so that $N_u'(T)$ and $N_u(T)$ will tend to be approximately equal. The following result makes this precise.

<u>THEOREM 13.17</u> With the above notation let $\{u_T\}$ be constants such that $P\{\xi(0) > u_T\} \to 0$, and that $T\mu(= T\mu(u_T)) \to \tau > 0$ as $T \to \infty$. Suppose that $E(N_u'(1))$ is finite for each u and that $E(N_u'(1)) \sim \mu(u)$ as $u \to \infty$. Then, writing $u_T = u$, $E(|N_u'(T) - N_u(T)|) \to 0$. If also the conditions of Theorem 13.15 hold (so that $P\{N_u(T) = r\} \to e^{-\tau}\tau^r/r!$) it follows that $P\{N_u'(T) = r\} \to e^{-\tau}\tau^r/r!$.

<u>PROOF</u> As noted above, $N_u'(T) \geq N_u(T) - 1$, and it is clear, moreover, that if $N_u'(T) = N_u(T) - 1$, then $\xi(T) > u$. Hence

$$E(|N_u'(T) - N_u(T)|) = E(N_u'(T) - N_u(T)) + 2P\{N_u'(T) = N_u(T) - 1\}$$
$$\leq TE(N_u'(1)) - \mu T + 2P\{\xi(T) > u\},$$

which tends to zero as $T \to \infty$ since $P\{\xi(T) > u_T\} = P\{\xi(0) > u_T\} \to 0$ and $TE(N_{u_T}'(1)) - \mu T = \mu T[(1 + o(1)) - 1] \to 0$, so that the first part of the theorem follows. The second part now follows immediately since the integer-valued r.v. $N_u'(T) - N_u(T)$ tends to zero in probability, giving $P\{N_u'(T) \neq N_u(T)\} \to 0$ and hence $P\{N_u'(T) = r\} - P\{N_u(T) = r\} \to 0$ for each r. $\square$

Now write $M^{(r)}(T)$ for the r-th largest local maximum in the interval $(0,T)$. Since the events $\{M^{(r)}(T) \le u\}$, $\{N_u'(T) < r\}$ are identical we obtain the following corollary.

<u>COROLLARY 13.18</u> Under the conditions of the theorem

$$P\{M^{(r)}(T) \le u_T\} \to e^{-\tau} \sum_{s=1}^{r-1} \tau^s/s! \quad . \qquad \square$$

As a further corollary we obtain the limiting distribution of $M^{(r)}(T)$ in terms of that for $M(T)$.

<u>COROLLARY 13.19</u> Suppose that $P\{a_T(M(T)-b_T) \le x\} \to G(x)$ and that the conditions of Theorem 13.8 hold with $u_T = x/a_T + b_T$ for each real x (and $\psi = \mu$). Suppose also that $E(N_u'(1)) \sim E(N_u(1))$ as $u \to \infty$. Then

(13.41) $$P\{a_T(M^{(r)}(T)-b_T) \le x\} \to G(x) \sum_{s=0}^{r-1} [-\log G(x,]^s/s!$$

where $G(x) > 0$ (and zero if $G(x) = 0$).

<u>PROOF</u> This follows from Corollary 13.18 by writing $G(x) = e^{-\tau}$ since Theorem 13.8 implies that $T\mu \to \tau$. $\square$

Note that, by Lemma 9.4(i), for a stationary *normal* process with finite second and fourth spectral moments $E(N_u'(1)) \sim \mu$ so that Theorem 13.17 and its corollaries apply.

The relation (13.41) gives the asymptotic distribution of the r-th largest local maximum $M^{(r)}(T)$ as a corollary of Theorem 13.17. Further it is clearly possible to generalize Theorem 13.17 to give "full Poisson convergence" for the point process of local maxima of height above u and indeed to generalize Theorem 9.5 and obtain joint distributions of heights and positions of local maxima in this general situation.

Interpretation of the function $\psi(u)$

The function $\psi(u)$ used throughout this chapter describes the tail of the distribution of the maximum $M(h)$ in a fixed interval h, in the sense of (13.9), viz.,

$$P\{M(h) > u\} \sim h\psi(u) \quad \text{for} \quad 0 < h \le h_0.$$

We have seen how ψ may be calculated for particular cases — as $\psi(u) = \mu(u)$ for processes with a finite upcrossing intensity $\mu(u)$ and as $\psi(u) = K\phi(u)u^{(2/\alpha)-1}$ for normal processes satisfying (12.1). Berman (1979) has recently considered another general method for obtaining ψ (or at least many of its properties) based on the asymptotic distribution of the amount of time spent above a high level.

Specifically Berman considers the time $L_t(u)$ which a stationary process spends above the level u in the interval $(0,t)$ and proves the basic result

$$\lim_{u\to\infty} \frac{P\{vL_t(u) > x\}}{E(vL_t(u))} \to -\Gamma'(x),$$

at all continuity points $x > 0$ of Γ' (under given conditions). Here $v = v(u)$ is a certain function of u and $\Gamma(x)$ is an absolutely continuous non-increasing function with Radon-Nikodym derivative Γ', and t is fixed.

While this result does not initially apply at $x = 0$, it is extended to so apply giving, since the events $\{M(t) > u\}$, $\{L_t(u) > 0\}$ are equivalent,

$$P\{M(t) > u\} \sim -\Gamma'(0)\ E(vL_t(u))$$
$$= -\Gamma'(0)\ vt(1-F(u)),$$

where F is the marginal d.f. of the process, since it is very easily shown that $E(L_t(u)) = t(1-F(u))$. Hence we may — under the stated condition — obtain ψ as

$$\psi(u) = -\Gamma'(0)\, v(u)(1 - F(u)).$$

It is required in this approach that F have such a form that it belongs to the domain of attraction of the Type I extreme value distribution and it follows (though not immediately) that $M(h)$ has a Type I limit so that (e.g. from the theory of this chapter) a limiting distribution for $M(T)$ as $T \to \infty$ must (under appropriate condition) also be of Type I. However a number of important cases are covered in this approach including stationary normal processes, certain Markov Processes, and so-called χ^2-processes. Further the approach gives considerable insight into the central ideas governing extremal properties.

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In this work we explore extremal and related theory for continuous parameter stationary processes. A general theory extending that for the sequence case, described in Chapter 2 of Part 1, is obtained, based on dependence conditions closely related to those used there for sequences. In particular, a general form of Gnedenko's Theorem is proved for the maximum

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$$M(T) = \sup\{\xi(t);\ 0 \le t \le T\},$$

where $\xi(t)$ is a stationary stochastic process satisfying appropriate regularity and dependence conditions. Cases where the process $\{\xi(t)\}$ is *normal* are discussed in detail.

Related topics include point process properties of upcrossings of levels, sample path properties at such upcrossings, location of extremes, maxima and minima for two or more dependent processes, and properties of local extremes.

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